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COUNTABLE INJECTIVE MODULES ARE SIGMA INJECTIVE

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ABSTRACT. In this note we show that a countable injective module is Σ-injective and consequently a ring R is left noetherian if the category of left R-modules has a countable injective cogenerator. Our proof can be modified to establish the corresponding result for quasi-injective modules. We also give an example of a nonnoetherian commutative ring R such that the category of R-modules has a countable cogenerator.

We let R denote an arbitrary ring with identity and M a unital left R-module. Recall that M is injective if and only if for each left ideal I of R and each R-homomorphism $f: I \to M$ there is a $y \in M$ such that $f(r) = ry$ for all $r \in I$. If X is a subset of M, then $\mathfrak{l}_R(X)$ is the left ideal consisting of those $r \in R$ such that $rx = 0$ for all $x \in X$. Similarly if I is a subset of R, we let $\mathfrak{r}_M(I) = \{x \in M: Ix = 0\}$. If an arbitrary direct sum of copies of M is injective, then M is said to be Σ-injective. Faith [4] has shown that an injective module M is Σ-injective if and only if the ascending chain condition holds for the left annihilator ideals $\mathfrak{l}_R(X)$.

THEOREM. *A countable injective module is Σ-injective.*

PROOF. Let $y_1, y_2, \ldots, y_n, \ldots$ be an enumeration of the elements of the countable injective R-module M. Assume by way of contradiction that there exists a strictly ascending chain $I_1 \subset I_2 \subset \cdots \subset I_n \subset \ldots$ of left annihilator ideals. If we let $X_n = \mathfrak{r}_M(I_n)$, then $I_n = \mathfrak{l}_R(X_n)$ and in M we have the strictly descending chain $X_1 \supset X_2 \supset \cdots \supset X_n \ldots$. Moreover if $X = \bigcap_{n=1}^{\infty} X_n$, then $X = r_M(I)$ where $I = \bigcup_{n=1}^{\infty} I_n$. We now construct inductively a sequence $b_1, b_2, \ldots, b_n, \ldots$ in I and a corresponding sequence of R-homomorphisms $f_n: \sum_{i=1}^{n} Rb_i \to M$ with $f_n \subseteq f_{n+1}$ and $f_n(b_n) \neq b_n y_n$ for all n. For $n = 1$, we choose a $z_1 \in X_1$ such that $z_1 - y_1 \notin X$. Since $X = \mathfrak{r}_m(I)$ there is some $b_1 \in I$ such that $b_1(z_1 - y_1) \neq 0$ and thus the homomorphism $f_1: Rb_1 \to M$ given by right multiplication by z_1 has the property that $f_1(b_1) \neq b_1 y_1$. Now suppose we have found $b_1, \ldots, b_n$ and $f_1, \ldots, f_n$ with the desired properties. Since M is injective, there is a z_n in M such that $f_n(r) = rz_n$ for all r in the domain of f_n. For sufficiently large m, we have $b_1, \ldots, b_n$ in I_m and we select z_{n+1} in X_m such that $z_{n+1} + z_n - y_{n+1} \notin X$. Then there will

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exist some b_{n+1} in I such that $b_{n+1}(z_{n+1} + z_n - y_{n+1}) \neq 0$ and the map f_{n+1}: $\Sigma_{i=1}^{n+1} Rb_1 \to M$ given by right multiplication by $z_{n+1} + z_n$ has the required properties. Finally to obtain the desired contradiction we note that the supremum f of all the f_n's is a homomorphism from the left ideal $\Sigma_{i=1}^{\infty} Rb_i$ into M and therefore there is a $y \in M$ such that $f(r) = ry$ for all r in the domain of f. But this yields $b_n y = f(b_n) = f_n(b_n) \neq b_n y_n$ for all n, contrary to the fact that y must equal some y_n.

REMARK. The foregoing proof is but a slight modification of the argument given by Lawrence [6] to show that a countable self-injective ring is necessarily quasi-Frobenius. As in that paper, this argument can be generalized to show that if M is an injective r-module of regular cardinality m, then any well-ordered properly ascending chain in R of left annihilators of subsets of M must have length less than m.

Recall that M is a cogenerator if each left R-module can be imbedded as a submodule of a product of sufficiently many copies of M. Since it is easily seen that the left ideal I is the annihilator of a subset of M if (and only if) R/I can be imedded in a product of copies of M, every left ideal of R will be the annihilator of a subset of M provided the latter is a cogenerator. Thus we immediately have the following

COROLLARY 1. *If the category of left R-modules has a countable injective cogenerator, then R is left noetherian.*

Let J be the Jacobson radical of R. We call R semilocal if R/J is semisimple. For such a ring R we have only finitely many isomorphically distinct simple left R-modules $S_1, \ldots, S_n$ and as an injective cogenerator we have $E(S_1) \oplus \cdots \oplus E(S_n)$ where $E(S_i)$ is the injective envelope of S_i. Therefore from Corollary 1 we have the following result.

COROLLARY 2. *If R is semilocal and if the injective envelope of each simple left R-module is countable, then R is left noetherian.*

Since a nilideal in a left noetherian ring is nilpotent and a semiprimary ring is left artinian if and only if it is left noetherian, we can also make the following observation.

COROLLARY 3. *If R is a semilocal ring with nil-Jacobson radical and if the injective envelope of each simple left R-module is countable, then R is left artinian.*

Examples exist showing that "injective cogenerator" cannot be weakened to "cogenerator" in Corollary 1 and "semilocal" is an essential hypothesis in corollary 2. Indeed there exist countable, commutative, nonnoetherian rings R such that for each maximal ideal P of R the localization R_p is a rank one discrete valuation ring. For such a ring R, $E(S)$ will be countable for each simple R-module S (see [7, Theorem 3.11]) in spite of the fact that R is not noetherian. Moreover as noted in [2] such an R can be constructed in which exactly one maximal ideal fails to be finitely generated. Under these circumstances R can contain only countably many maximal ideals which in turn give rise to countably many isomorphically distinct simple R-modules $S_1, S_2, \ldots, S_n, \ldots$. Then the countable module $M = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n) \oplus \ldots$ is a cogenerator (see, for example, [**1**, 18.16]), but it is not injective by Corollary 1 since R is not noetherian.

Finally we wish to note that the proof of our theorem can easily be modified to yield the same conclusion for countable quasi-injective modules. Recall that M is quasi-injective if each homomorphism $f\colon N \to M$ with N a submodule of M extends to an endomorphism of M. It is not difficult to generalize a result of Fuchs [**5**] in order to show that M is quasi-injective if and only if it satisfies the following condition: If I is a left ideal of R and if $f\colon I \to M$ is an R-homomorphism with $\operatorname{Ker} f \supset \mathfrak{l}_R(F)$ for some finite subset F of M, then there is a $y \in M$ such that $f(r) = ry$ for all $r \in I$. Then armed with the characterization of Σ-quasi-injective modules given in [**3**], one can readily carry out the desired proof that countable quasi-injective modules are Σ-quasi-injective.

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