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TOPOLOGICALLY UNREALIZABLE AUTOMORPHISMS OF FREE GROUPS

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ABSTRACT. Let $\phi: F \to F$ be an automorphism of a finitely generated free group. It has been conjectured (I heard it from Peter Scott) that the fixed subgroup of ϕ is always finitely generated. This is known to be so if ϕ has finite order [1], or if ϕ is realizable by a homeomorphism of a compact 2-manifold with boundary [2]. Here we give examples of automorphisms ϕ, no power of which is topologically realizable on any 2-manifold; perhaps the simplest is the automorphism of the free group of rank 3, given by $\phi(x) = y$, $\phi(y) = z$, $\phi(z) = xy$.

1. PV-matrices and automorphisms. By a PV-matrix is meant an $n \times n$ integer matrix of determinant ± 1, having one eigenvalue, λ_1, of absolute value greater than 1, and $n - 1$ eigenvalues of absolute value less than 1. The terminology "PV" is used because λ_1 is a Pisot-Vijayaraghavan number [3].

A PV-automorphism of a free abelian group of rank n is an automorphism whose matrix is a PV-matrix, (PV-ness is independent of the basis). A PV-automorphism of a free (nonabelian) group, is an automorphism whose abelianization is PV.

1.1. If M is an $n \times n$ PV-matrix and $n \geqslant 3$, then, since $|\lambda_1\lambda_2 \ldots \lambda_n| = 1$, no eigenvalue λ_i is the inverse of any λ_j.

1.2. If M is a PV-matrix, then every positive integral power M^k is a PV-matrix, since the eigenvalues of M^k are the kth powers of those of M.

A simple 3×3 example of a PV-matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

with eigenvalues approximately

$$\lambda_1 = 1.3247, \qquad \lambda_2, \lambda_3 = -0.6624 \pm 0.5623\sqrt{-1}\,.$$

Correspondingly, $\phi(x) = y$, $\phi(y) = z$, $\phi(z) = xy$ describes a PV-automorphism of the free group with basis $\{x, y, z\}$.

2. Eigenvalues of automorphisms of a 2-manifold.

2.1. Let $h: T \to T$ be an orientation-preserving homeomorphism of a closed, orientable 2-manifold onto itself. Then the eigenvalues of the homology map

$$h_*: H_1(T) \to H_1(T)$$

occur in inverse pairs; that is, they can be listed $\lambda_1, \lambda_2, \ldots, \lambda_g, \lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_g^{-1}$.

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The reason is that h_* preserves an alternating bilinear form, the intersection number, and thus h_* is symplectic, and every symplectic matrix is similar to its inverse. (Cf. Exercise 1, §6.9, p. 377 in [4].)

2.2. Let $h\colon T \to T$ be an orientation-preserving homeomorphism of a compact, orientable, connected 2-manifold, with $\beta \geq 1$ boundary components, onto itself, such that h maps each boundary component to itself. Then the eigenvalues of the homology map

$$h_*\colon H_1(T) \to H_1(T)$$

consist of two sorts: There are $\beta - 1$ eigenvalues $= 1$, and the remaining eigenvalues occur in inverse pairs.

The reason for this is that the subgroup S of $H_1(T)$ generated by the boundary components is a free abelian direct summand of rank $\beta - 1$; and h_* maps S to itself by the identity. On the quotient by S, h_* is the homology map on the closed manifold obtained by capping off the boundary components with 2-cells, and to this we apply 2.1.

3. PV-automorphisms are not realizable.

THEOREM. *Let $\phi\colon F \to F$ be a PV-automorphism of a free group of infinite rank $n \geq 3$. Then, for every integer $k \geq 1$, ϕ^k is not realizable, as the automorphism on fundamental group, by any homeomorphism $h\colon T \to T$ of any 2-manifold T.*

PROOF. There are two cases, T orientable or not.

Orientable case. If T is orientable, then h^2 is orientation preserving and permutes the boundary components of T; this permutation has some finite order q, so that h^{2q} is orientation preserving and maps each boundary component to itself. The homomorphism

$$h_*^{2q}\colon H_1(T) \to H_1(T)$$

is the abelianization of ϕ^{2kq}. This is a PV-automorphism by 1.2, and thus none of the eigenvalues of h_*^{2q} is the inverse of any other by 1.1. This contradicts 2.2.

Nonorientable case. If T is nonorientable, let $T' \to T$ be its orientable double cover. The homeomorphism $h\colon T \to T$ lifts to a homeomorphism $h'\colon T' \to T'$. There is a transfer homomorphism

$$\tau\colon H_1(T) \to H_1(T')$$

such that $\tau \circ h_* = h'_* \circ \tau$, and such that the composition

$$H_1(T) \to H_1(T') \to H_1(T)$$

is multiplication by 2.

If we take the coefficient group to be the field of rational numbers–this does not change any argument on eigenvalues–then τ embeds $H_1(T)$ as a subspace of $H_1(T')$ which is invariant under h'_* and on which h'_* is isomorphic to h_*.

Now, as in the orientable case, there is some positive integer q such that $(h')^{2q}$ is orientation preserving and maps each boundary component of T' to itself. The list of eigenvalues of $(h'_*)^{2q}$ includes, by the transfer argument, the eigenvalues of h_*^{2q}. The latter are eigenvalues of the abelianization of ϕ^{2kq}, which, as before, do not include any inverse pairs.

There are n eigenvalues of h_*^{2q}, since the rank of $H_1(T)$ is the rank of F, which is n. A Euler characteristic argument shows that the rank of $H_1(T')$ is $2n - 1$. Therefore, there are not enough additional eigenvalues of $(h'_*)^{2q}$ to make up a set of eigenvalues satisfying 2.2.

4. Comments.

4.1. Suppose that $\phi: F \to F$ is a PV-automorphism and that $S \subset F$ is a subgroup of finite index with $\phi(S) = S$. I suspect that $\phi|S$ is not realizable by a surface homeomorphism and that this can be proved by examining eigenvalues. If S is a normal subgroup and ϕ induces the identity on $G = F/S$, then ϕ determines a ZG-automorphism on the abelianization of S. Can this automorphism have a symmetric set of eigenvalues (satisfying 2.2)? This seems to involve the question: What does the fact that F is free imply about the structure of the abelianization of S as a ZG-module?

4.2. Every automorphism $\phi: F \to F$ of a free group of rank n, whose abelianization has determinant $+1$, leaves something fixed modulo the $(n + 1)$st term in the lower central series.

PROOF. Define

$$F_1 = F, \qquad F_{k+1} = [F, F_k].$$

Then the quotients of the lower central series $L_k = F_k/F_{k+1}$ form a free Lie algebra over Z, and ϕ induces automorphisms $\phi_k: L_k \to L_k$. Tensor with the complex numbers $\mathbf{C}$. Then ϕ_1 has eigenvalues $\lambda_1, \ldots, \lambda_n$, and corresponding eigenvectors $\xi_1, \ldots, \xi_n$ in $L_1 \otimes \mathbf{C}$. However, we need to have ξ_{n-1} and ξ_n linearly independent, so that if $\lambda_{n-1} = \lambda_n$, it may be necessary to change the defining equation for ξ_n from $\phi_1(\xi_n) = \lambda_n \xi_n$ to $\phi_1(\xi_n) = \lambda_n \xi_n + \xi_{n-1}$.

Then the element

$$\eta = [\xi_1, [\ldots, \xi_\eta]] \quad \text{in } L_n \otimes \mathbf{C}$$

is nonzero and has the property that

$$\phi_n(\eta) = \lambda_1 \ldots \lambda_n \eta = \eta.$$

Thus ϕ_n has an eigenvalue 1, and therefore has an *integral* eigenvector θ corresponding to the eigenvalue 1. Then θ is represented by $w \in F_n - F_{n+1}$ such that $\phi(w) \equiv w$ modulo F_{n+1}.

For example, taking ϕ to be the PV-automorphism described at the end of §1, both

$$[x, [y, z]][x, [x, z]][y, [y, x]][z, [z, y]]$$

and

$$[x, [y, z]][x, [x, y]][x, [x, z]][y, [y, z]][z, [z, x]][z, [z, y]]$$

are fixed modulo F_4.

Hence, it is at least conceivable that this automorphism leaves something in F_3 fixed. My conjecture, which I cannot prove, is that the fixed subgroup of every PV-automorphism is trivial.

References

1. J. L. Dyer and G. P. Scott, *Periodic automorphisms of free groups*, Comm. Algebra **3** (1975), 195–201.

2. W. Jaco and P. B. Shalen, *Surface homeomorphisms and periodicity*, Topology **16** (1977), 347–367.

3. J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge Univ. Press, New York, 1957.

4. N. Jacobson, *Basic algebra*. I, Freeman, San Francisco, Calif., 1974.

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