VIEW-OBSTRUCTION PROBLEMS. II

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Abstract. Let $S^n$ denote the region $0 < x_i < \infty$ ($i = 1, 2, \ldots, n$) of $n$-dimensional Euclidean space $E^n$. Suppose $C$ is a closed convex body in $E^n$ which contains the origin as an interior point. Define $\alpha C$ for each real number $\alpha > 0$ to be the magnification of $C$ by the factor $\alpha$ and define $C + (m_1, \ldots, m_n)$ for each point $(m_1, \ldots, m_n)$ in $E^n$ to be the translation of $C$ by the vector $(m_1, \ldots, m_n)$. Define the point set $\Delta(C, \alpha)$ by $\Delta(C, \alpha) = \{\alpha C + (m_1, \ldots, m_n) : m_1, \ldots, m_n$ nonnegative integers$. The view-obstruction problem for $C$ is the problem of finding the constant $K(C)$ defined to be the lower bound of those $\alpha$ such that any half-line $L$ given by $x_i = a_i t$ ($i = 1, 2, \ldots, n$), where the $a_i$ ($1 \leq i < n$) are positive real numbers, and the parameter $t$ runs through $[0, \infty)$, intersects $\Delta(C, \alpha)$.

The paper considers the case where $C$ is the $n$-dimensional cube with side 1, and in this case the constant $K(C)$ is known for $n < 3$. The paper gives a new proof for the case $n = 3$. Unlike earlier proofs, this one could be extended to study the cases with $n > 4$.

1. Introduction. The view-obstruction problems defined in the abstract were first introduced in [2]. In this paper we only consider the case where the closed convex body $C$ in $E^n$ is the $n$-dimensional cube with side 1. We use the notation $X(n)$ for the constant $K(C)$ in this case.

For any real number $x$, let $||x||$ denote the distance from $x$ to the nearest integer. The evaluation of $X(n)$ can be thought of as a problem in Diophantine approximation, since we have

$$\frac{1}{2}X(n) = \sup_{0 < x < 1} \min_{1 \leq i < n} \max_{1 \leq i < n} \|w_i x - \frac{1}{2}\|,$$

where the supremum is taken over all $n$-tuples $w_1, \ldots, w_n$ of positive integers. If we define

$$(1) \quad \kappa(n) = \inf_{0 < x < 1} \max_{1 \leq i < n} \min_{1 \leq i < n} \|w_i x\|,$$

where the infimum is taken over all $n$-tuples $w_1, \ldots, w_n$ of positive integers, then since $\|w_i x\| = \frac{1}{2} - \|w_i x - \frac{1}{2}\|$, we have $X(n) = 1 - 2\kappa(n)$ for each $n > 2$. It will be convenient in the rest of the paper to concentrate on the problem of evaluating $\kappa(n)$.

The problem of evaluating $X(n)$ is equivalent to the following: Suppose the unit cube in $E^n$ has faces which reflect a certain particle, and consider any motion of the particle, starting in a corner of the cube and not entirely contained in a hyperplane of dimension $n - 1$. What is the side length of the largest subcube,
centered in the unit cube, with the property that there exists such a motion of the 
particle which does not intersect the subcube? Plainly the largest such side length is 
\( \lambda(n) \).

The corresponding problem, if the condition that the particle start in a corner is 
omitted, can be treated by methods entirely different from those in this paper. This 
has been done by I. J. Schoenberg [4], who solved this problem in every dimension; 
he showed that the largest subcube in dimension \( n \) has side \( 1 - n^{-1} \).

The natural conjecture for the value of \( \lambda(n) \) is \( (n - 1)/(n + 1) \) (as stated in [2, p. 
166]). This is because Dirichlet's box principle gives \( \max_{0 \leq x \leq 1} \min_{1 \leq i \leq n} \|ix\| = 
1/(n + 1) \), so \( \kappa(n) < 1/(n + 1) \), and it is reasonable to conjecture that equality 
holds. It is this conjecture that is proved in this paper for \( n < 3 \).

The case \( n = 2 \) is very simple. The case \( n = 3 \) is considerably more complicated, 
but several proofs have previously been published (Betke and Wills [1], Cusick [2, 
3]), and another one is given here. The proof in this paper, unlike the earlier proofs, 
has the advantage that it can be extended to study the cases with \( n > 4 \); however, 
the argument is then no longer elementary. The author hopes to return to this 
question elsewhere.

2. Another proof that \( \kappa(3) = \frac{1}{2} \). By (1), in order to show that \( \kappa(n) = 1/(n + 1) \) it is 
enough to prove that given any \( n \)-tuple \( w_1, \ldots, w_n \) of positive integers with the 
property that, for any integers \( m \) and \( q \),

\[
\|w_i q / m\| < 1/ (n + 1) \quad \text{for some } i, 1 < i < n,
\]

there exists some pair \( m, q \) such that (2) does not hold if \( < \) is replaced by \( < \).

If we assume (as we may with no loss of generality) that \( w_1, \ldots, w_n \) have no 
common prime factor, then we would expect that there are only finitely many 
\( n \)-tuples \( w_1, \ldots, w_n \) such that (2) holds for any \( m \) and \( q \). Further, we might hope 
that by considering only finitely many values of \( m \), we could identify all of these 
\( n \)-tuples, and so reduce the determination of \( \kappa(n) \) to a finite calculation. It is easy to 
carry out this procedure when \( n = 2 \), and so prove \( \kappa(2) = \frac{1}{2} \). In the remaining 
portion of this paper, we show that the procedure can also be successfully carried 
out when \( n = 3 \).

For the rest of this section, we take \( n = 3 \) and suppose \( w_1, w_2, w_3 \) is a triple of 
integers, having no common prime factor, such that (2) holds for any integers \( m 
and \( q \). Our goal is to show that we can always find a pair of integers \( m \) and \( q \) such 
that

\[
\min_{1 < i < 3} \|w_i q / m\| > \frac{1}{4}.
\]

If \( w \) is odd, then \( \|\frac{w}{3}\| = \frac{1}{2} \), so we can assume that at least one of the \( w_i \) is even. It 
is easy to prove that (3) holds if exactly two of the \( w_i \) are even. First suppose that 
\( w_1 = 2^a a, w_2 = 2^b b \) and \( w_3 = c \), where \( a, b, c \) are odd and \( k \) is \( > 1 \). If we take 
\( m = 2^{k+1} \) and choose \( q \) to be any odd integer such that \( qc \equiv 2^k + 1 \mod 2^{k+1} \), 
then (3) holds. Next suppose that \( w_1 = 2^{i+k} a, w_2 = 2^b b \) and \( w_3 = c \), where \( a, b, c 
are odd and \( j, k \) are \( > 1 \). We take \( m = 2^{j+k+1} \) and will take \( q \) to be odd, so
In order to specify \( q \), we first choose an odd \( q_0 \) such that \( bq_0 = t \mod 2^{j+1} \), where \( t \) is an odd integer satisfying \( \|t/2^{j+1}\| > \frac{1}{4} \). We define \( q \) to be \( q_0 + 2^{j+1}r \), where \( r \) is chosen so that \( \|w_r/2^m\| > \frac{1}{4} \) (such a choice of \( r \) is possible since changing \( r \) by 1 changes \( \|w_r/2^m\| \) by \( \|c/2^k\| \)). With this choice of \( q \), (3) holds.

Now we suppose that exactly one of the \( w_i \) is even, say \( w_1 = 2^ka, w_2 = b, w_3 = c \), where \( a, b, c \) are odd and \( k \) is \( > 1 \). For this case we need the following elementary lemma.

**Lemma 1.** For \( u \) any odd integer and \( n \) any integer \( > 3 \), define \( S_n(u) = S(u) = \{ \text{least positive residues mod } 2^n \text{ of odd } t \text{ satisfying } \|tu/2^n\| > \frac{1}{2} \} \). Then for any pair \( u, v \) such that \( 1 < u, v < 2^{n-1} \), we have

\[
S(u) \cup S(v) = \{ \text{all odd } t \text{ mod } 2^n \}
\]

if and only if \( u + v \equiv 0 \mod 2^{n-1} \).

**Proof.** The "if" part of the lemma is clear, since \( u + v \equiv 0 \mod 2^{n-1} \) and \( 1 < u, v < 2^{n-1} \) means \( u + v = 2^{n-1} \), so \( \|tu/2^n\| = \|(2^{n-1} - t)v/2^n\| \). Since \( t \) belongs to \( S(u) \) if and only if \( 2^{n-1} - t \) does not belong to \( S(u) \), we have (4).

To prove the "only if" part of the lemma, it is enough to show that \( S(u) = S(v) \) cannot happen if \( u \neq v \) and \( 1 < u, v < 2^{n-1} \); for if (4) holds with \( 1 < u, v < 2^{n-1} \), then \( S(2^{n-1} - v) = S(u) \). Define

\[
M = \{ \text{odd } m \text{ satisfying } 2^{n-2} < m < 3 \cdot 2^{n-2} \},
\]

so \( r \) is in \( S(u) \) if and only if \( ru \equiv m \mod 2^n \) for some \( m \) in \( M \). Thus \( S(u) = S(v) \) means that the set \( M \) is unchanged when the elements of \( M \) are multiplied by \( u^{-1}v \) and reduced mod \( 2^n \); we use the notation \( u^{-1}vM = M \) for this. We prove that if \( x \) is any integer such that \( xM = M \), then \( x \equiv \pm 1 \mod 2^n \). This will complete the proof of the lemma; for then \( S(u) = S(v) \) implies either \( u^{-1}v \equiv -1 \mod 2^n \) (so \( u + v \equiv 0 \mod 2^n \), which is impossible if \( 1 < u, v < 2^{n-1} \)) or \( u^{-1}v \equiv 1 \mod 2^n \) (so \( u = v \)).

So we suppose \( xM = M \) with \( 1 < x < 2^n \). Clearly this implies \( x^iM = M \) for each \( i = 1, 2, \ldots \). Let \( d \) be the order of \( x \mod 2^n \); note \( d \) is even since \( d > 1 \) and \( d \) divides \( \varphi(2^n) = 2^{n-1} \). We have \( (x^{d/2})^2 \equiv 1 \mod 2^n \), and the roots of \( y^2 \equiv 1 \mod 2^n \) for \( n > 3 \) are \( y \equiv 1, -1, 2^{-1} + 1 \) or \( 2^{-1} - 1 \). We cannot have \( x^{d/2} \equiv 1 \mod 2^n \) (this contradicts the definition of \( d \)) or \( x^{d/2} \equiv 2^{-1} \pm 1 \mod 2^n \) (for then \( x^{d/2}M = M \), but the element \( x^{d/2}(2^{-1} \pm 1) \equiv 1 \) is not in \( M \)-contradiction). Hence \( x^{d/2} \equiv -1 \mod 2^n \); if \( d \) is divisible by \( 4 \) this is impossible because \( y^2 \equiv -1 \mod 2^n \) has no solutions. Hence \( d = 2 \), so \( x \equiv -1 \mod 2^n \) and the proof is complete.

Now we turn to the proof of (3) for \( w_1 = 2^ka, w_2 = b, w_3 = c \). We will choose \( q \) odd and \( m \) equal to either \( 2^{k+1} \) or \( 2^{k+2} \), so we may assume without loss of generality that \( 1 < b, c < 2^{k+1} \). First suppose that \( m = 2^{k+2} \). If we can find an integer \( q \) such that \( q \) belongs to both \( S_{k+2}(b) \) and \( S_{k+2}(c) \) (using the notation of Lemma 1), then (3) holds. If no such \( q \) exists, then by Lemma 1 with \( n = k + 2 \) we have \( b + c \equiv 0 \mod 2^{k+1} \). This means \( S_{k+1}(b) = S_{k+1}(c) \), so if we choose \( q \) to be any integer in \( S_{k+1}(b) \) and take \( m = 2^{k+1} \), then (3) holds. This finishes the proof that \( \kappa(3) = \frac{1}{4} \).
REFERENCES


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