PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

VIEW-OBSTRUCTION PROBLEMS. II

T. W. CUSICK

ABSTRACT. Let S^n denote the region $0 < x_i < \infty$ $(i = 1, 2, \ldots, n)$ of n-dimensional Euclidean space E^n. Suppose C is a closed convex body in E^n which contains the origin as an interior point. Define αC for each real number $\alpha > 0$ to be the magnification of C by the factor α and define $C + (m_1, \ldots, m_n)$ for each point $(m_1, \ldots, m_n)$ in E^n to be the translation of C by the vector $(m_1, \ldots, m_n)$. Define the point set $\Delta(C, \alpha)$ by $\Delta(C, \alpha) = \{\alpha C + (m_1 + \frac{1}{2}, \ldots, m_n + \frac{1}{2})$: $m_1, \ldots, m_n$ nonnegative integers$\}$. The *view-obstruction problem for* C is the problem of finding the constant $K(C)$ defined to be the lower bound of those α such that any half-line L given by $x_i = a_i t$ $(i = 1, 2, \ldots, n)$, where the a_i $(1 < i < n)$ are positive real numbers, and the parameter t runs through $[0, \infty)$, intersects $\Delta(C, \alpha)$.

The paper considers the case where C is the n-dimensional cube with side 1, and in this case the constant $K(C)$ is known for $n < 3$. The paper gives a new proof for the case $n = 3$. Unlike earlier proofs, this one could be extended to study the cases with $n > 4$.

1. Introduction. The view-obstruction problems defined in the abstract were first introduced in [2]. In this paper we only consider the case where the closed convex body C in E^n is the n-dimensional cube with side 1. We use the notation $\lambda(n)$ for the constant $K(C)$ in this case.

For any real number x, let $\|x\|$ denote the distance from x to the nearest integer. The evaluation of $\lambda(n)$ can be thought of as a problem in Diophantine approximation, since we have

$$\tfrac{1}{2}\lambda(n) = \sup \min_{0<x<1} \max_{1<i<n} \|w_i x - \tfrac{1}{2}\|,$$

where the supremum is taken over all n-tuples $w_1, \ldots, w_n$ of positive integers. If we define

$$\kappa(n) = \inf \max_{0<x<1} \min_{1<i<n} \|w_i x\|, \tag{1}$$

where the infimum is taken over all n-tuples $w_1, \ldots, w_n$ of positive integers, then since $\|w_i x\| = \frac{1}{2} - \|w_i x - \frac{1}{2}\|$, we have $\lambda(n) = 1 - 2\kappa(n)$ for each $n > 2$. It will be convenient in the rest of the paper to concentrate on the problem of evaluating $\kappa(n)$.

The problem of evaluating $\lambda(n)$ is equivalent to the following: Suppose the unit cube in E^n has faces which reflect a certain particle, and consider any motion of the particle, *starting in a corner* of the cube and not entirely contained in a hyperplane of dimension $n - 1$. What is the side length of the largest subcube,

Received by the editors January 25, 1980.
1980 *Mathematics Subject Classification*. Primary 10F10; Secondary 10E99.
© 1982 American Mathematical Society
0002-9939/82/0000-0006/$02.00

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

centered in the unit cube, with the property that there exists such a motion of the particle which does not intersect the subcube? Plainly the largest such side length is $\lambda(n)$.

The corresponding problem, if the condition that the particle start in a corner is omitted, can be treated by methods entirely different from those in this paper. This has been done by I. J. Schoenberg [**4**], who solved this problem in every dimension; he showed that the largest subcube in dimension n has side $1 - n^{-1}$.

The natural conjecture for the value of $\lambda(n)$ is $(n - 1)/(n + 1)$ (as stated in [**2**, p. 166]). This is because Dirichlet's box principle gives $\max_{0 < x < 1} \min_{1 \leqslant i \leqslant n} \|ix\| = 1/(n + 1)$, so $\kappa(n) \leqslant 1/(n + 1)$, and it is reasonable to conjecture that equality holds. It is this conjecture that is proved in this paper for $n \leqslant 3$.

The case $n = 2$ is very simple. The case $n = 3$ is considerably more complicated, but several proofs have previously been published (Betke and Wills [**1**], Cusick [**2**, **3**]), and another one is given here. The proof in this paper, unlike the earlier proofs, has the advantage that it can be extended to study the cases with $n \geqslant 4$; however, the argument is then no longer elementary. The author hopes to return to this question elsewhere.

2. Another proof that $\kappa(3) = \frac{1}{4}$. By (1), in order to show that $\kappa(n) = 1/(n + 1)$ it is enough to prove that given any n-tuple $w_1, \ldots, w_n$ of positive integers with the property that, for any integers m and q,

$$(2) \qquad \|w_i q/m\| \leqslant 1/(n + 1) \quad \text{for some } i,\ 1 \leqslant i \leqslant n,$$

there exists some pair m, q such that (2) does not hold if $\leqslant$ is replaced by $<$.

If we assume (as we may with no loss of generality) that $w_1, \ldots, w_n$ have no common prime factor, then we would expect that there are only finitely many n-tuples $w_1, \ldots, w_n$ such that (2) holds for any m and q. Further, we might hope that by considering only finitely many values of m, we could identify all of these n-tuples, and so reduce the determination of $\kappa(n)$ to a finite calculation. It is easy to carry out this procedure when $n = 2$, and so prove $\kappa(2) = \frac{1}{3}$. In the remaining portion of this paper, we show that the procedure can also be successfully carried out when $n = 3$.

For the rest of this section, we take $n = 3$ and suppose w_1, w_2, w_3 is a triple of integers, having no common prime factor, such that (2) holds for any integers m and q. Our goal is to show that we can always find a pair of integers m and q such that

$$(3) \qquad \min_{1 \leqslant i \leqslant 3} \|w_i \tfrac{q}{m}\| \geqslant \tfrac{1}{4}.$$

If w is odd, then $\|\frac{w}{2}\| = \frac{1}{2}$, so we can assume that at least one of the w_1 is even. It is easy to prove that (3) holds if exactly two of the w_i are even. First suppose that $w_1 = 2^k a$, $w_2 = 2^k b$ and $w_3 = c$, where a, b, c are odd and k is $\geqslant 1$. If we take $m = 2^{k+1}$ and choose q to be any odd integer such that $qc \equiv 2^k + 1 \bmod 2^{k+1}$, then (3) holds. Next suppose that $w_1 = 2^{j+k}a$, $w_2 = 2^k b$ and $w_3 = c$, where a, b, c are odd and j, k are $\geqslant 1$. We take $m = 2^{j+k+1}$ and will take q to be odd, so

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$\|w_1 q/m\| = \frac{1}{2}$. In order to specify q, we first choose an odd q_0 such that $bq_0 \equiv t \bmod 2^{j+1}$, where t is an odd integer satisfying $\|t/2^{j+1}\| \geqslant \frac{1}{4}$. We define q to be $q_0 + 2^{j+1}r$, where r is chosen so that $\|w_3 \frac{q}{m}\| \geqslant \frac{1}{4}$ (such a choice of r is possible since changing r by 1 changes $\|w_3 \frac{q}{m}\|$ by $\|c/2^k\|$). With this choice of q, (3) holds.

Now we suppose that exactly one of the w_i is even, say $w_1 = 2^k a$, $w_2 = b$, $w_3 = c$, where a, b, c are odd and k is $\geqslant 1$. For this case we need the following elementary lemma.

LEMMA 1. *For u any odd integer and n any integer $\geqslant 3$, define $S_n(u) = S(u) = \{$least positive residues $\bmod 2^n$ of odd t satisfying $\|tu/2^n\| > \frac{1}{4}\}$. Then for any pair u, v such that $1 \leqslant u, v < 2^{n-1}$, we have*

$$S(u) \cup S(v) = \{\text{all odd } t \bmod 2^n\} \tag{4}$$

if and only if $u + v \equiv 0 \bmod 2^{n-1}$.

PROOF. The "if" part of the lemma is clear, since $u + v \equiv 0 \bmod 2^{n-1}$ and $1 \leqslant u, v < 2^{n-1}$ means $u + v = 2^{n-1}$, so $\|tu/2^n\| = \|(2^{n-1} - t)v/2^n\|$. Since t belongs to $S(u)$ if and only if $2^{n-1} - t$ does not belong to $S(u)$, we have (4).

To prove the "only if" part of the lemma, it is enough to show that $S(u) = S(v)$ cannot happen if $u \neq v$ and $1 \leqslant u, v < 2^{n-1}$; for if (4) holds with $1 \leqslant u, v < 2^{n-1}$, then $S(2^{n-1} - v) = S(u)$. Define

$$M = \{\text{odd } m \text{ satisfying } 2^{n-2} < m < 3 \cdot 2^{n-2}\},$$

so r is in $S(u)$ if and only if $ru \equiv m \bmod 2^n$ for some m in M. Thus $S(u) = S(v)$ means that the set M is unchanged when the elements of M are multiplied by $u^{-1}v$ and reduced mod 2^n; we use the notation $u^{-1}vM = M$ for this. We prove that if x is any integer such that $xM = M$, then $x \equiv \pm 1 \bmod 2^n$. This will complete the proof of the lemma; for then $S(u) = S(v)$ implies either $u^{-1}v \equiv -1 \bmod 2^n$ (so $u + v \equiv 0 \bmod 2^n$, which is impossible if $1 \leqslant u, v < 2^{n-1}$) or $u^{-1}v \equiv 1 \bmod 2^n$ (so $u = v$).

So we suppose $xM = M$ with $1 < x < 2^n$. Clearly this implies $x^i M = M$ for each $i = 1, 2, \ldots$. Let d be the order of $x \bmod 2^n$; note d is even since $d > 1$ and d divides $\phi(2^n) = 2^{n-1}$. We have $(x^{d/2})^2 \equiv 1 \bmod 2^n$, and the roots of $y^2 \equiv 1 \bmod 2^n$ for $n \geqslant 3$ are $y \equiv 1, -1, 2^{n-1} + 1$ or $2^{n-1} - 1$. We cannot have $x^{d/2} \equiv 1 \bmod 2^n$ (this contradicts the definition of d) or $x^{d/2} \equiv 2^{n-1} \pm 1 \bmod 2^n$ (for then $x^{d/2}M = M$, but the element $x^{d/2}(2^{n-1} \pm 1) \equiv 1$ is not in M-contradiction). Hence $x^{d/2} \equiv -1 \bmod 2^n$; if d is divisible by 4 this is impossible because $y^2 \equiv -1 \bmod 2^n$ has no solutions. Hence $d = 2$, so $x \equiv -1 \bmod 2^n$ and the proof is complete.

Now we turn to the proof of (3) for $w_1 = 2^k a$, $w_2 = b$, $w_3 = c$. We will choose q odd and m equal to either 2^{k+1} or 2^{k+2}, so we may assume without loss of generality that $1 \leqslant b, c < 2^{k+1}$. First suppose that $m = 2^{k+2}$. If we can find an integer q such that q belongs to both $S_{k+2}(b)$ and $S_{k+2}(c)$ (using the notation of Lemma 1), then (3) holds. If no such q exists, then by Lemma 1 with $n = k + 2$ we have $b + c \equiv 0 \bmod 2^{k+1}$. This means $S_{k+1}(b) = S_{k+1}(c)$, so if we choose q to be any integer in $S_{k+1}(b)$ and take $m = 2^{k+1}$, then (3) holds. This finishes the proof that $\kappa(3) = \frac{1}{4}$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

References

1. U. Betke and J. M. Wills, *Untere Schranken für zwei diophantische Approximations-Funktionen*, Monatsh. Math. **76** (1972), 214–217.

2. T. W. Cusick, *View-obstruction problems*, Aequationes Math. **9** (1973), 165–170.

3. ______, *View-obstruction problems in n-dimensional geometry*, J. Combin. Theory Ser. A **16** (1974), 1–11.

4. I. J. Schoenberg, *Extremum problems for the motions of a billiard ball*. II. *The L_∞ norm*, Nederl. Akad. Wetensch. Proc. Ser. A **79** = Indag. Math. **38** (1976), 263–279.

Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use