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SUBHARMONIC FUNCTIONS OUTSIDE A COMPACT SET IN $\mathbf{R}^n$

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ABSTRACT. Let u be a subharmonic function defined outside a compact set in $\mathbf{R}^2$. Then u is of the form $u(x) = s(x) - \alpha \log|x|$ outside a disc where $s(x)$ is a nonconstant subharmonic function in $\mathbf{R}^2$ and $\alpha > 0$. Some applications and the analogues in $\mathbf{R}^n$, $n > 3$, are given.

1. Introduction. Let u be a subharmonic function defined outside a compact set in $\mathbf{R}^2$. We prove that outside a disc u is of the form $u(x) = s(x) - \alpha \log|x|$ where $s(x)$ is a nonconstant subharmonic function in $\mathbf{R}^2$ and $\alpha \geqslant 0$.

From the above decomposition, it is easily seen that $\mu(u) = \lim(M(r, u)/\log r)$ always exists where $M(r, u)$ is the mean value of u on $|x| = r$. We show that there exists a (nonharmonic) subharmonic function v in $\mathbf{R}^2$ such that $v = u$ outside a disc if and only if $\mu(u) > 0$. This result is implicit in M. Brelot [1] and it is proved here as a simple application of the above decomposition of u.

In particular, defining the order of u as ord u = ord s, we note that there exists a (nonharmonic) subharmonic function v in $\mathbf{R}^2$ such that $v = u$ outside a disc in the following two cases: (i) u is of nonintegral order, and (ii) u is lower bounded but not bounded.

Finally we state some of the analogous results in $\mathbf{R}^n$, $n > 3$.

2. Subharmonic functions outside a disc in $\mathbf{R}^2$.

THEOREM 1. *Suppose that u is a subharmonic function defined outside a compact set in $\mathbf{R}^2$. Then there exist a nonconstant subharmonic function s in $\mathbf{R}^2$ and a constant $\alpha > 0$ such that $u(x) = s(x) - \alpha \log|x|$ outside a disc.*

PROOF. Let u be finite continuous in a neighbourhood of $|x| = R > 1$ and subharmonic in $|x| > R_0$ ($R_0 < R$). Let $r > R$ and $D_r u$ denote the Dirichlet solution in $|x| < r$ with boundary value u.

Choose $\alpha \geqslant 0$ large so that $D_r u + \alpha \log r \geqslant u + \alpha \log R$ on $|x| = R$. This implies that $u(x) + \alpha \log|x| \leqslant D_r(u + \alpha \log|x|)$ in $R < |x| < r$.

Hence if $s(x)$ is the function $u(x) + \alpha \log|x|$ in $|x| > r$ extended by $D_r(u + \alpha \log|x|)$ in $|x| < r$, $s(x)$ is subharmonic in $\mathbf{R}^2$ and $u(x) = s(x) - \alpha \log|x|$ outside a disc. Moreover s can always be chosen as a nonconstant function since α is an arbitrary large positive number.

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REMARKS. (1) Since $\lim(M(r, s)/\log r)$ exists and equals the total mass associated with s in the local Riesz representation, $\mu(u) = \lim(M(r, u)/\log r)$ always exists and $-\infty < \mu(u) \leqslant \infty$.

(2) When $\mu(u) < \infty$, the least harmonic majorant of u outside a disc is of the form $\mu(u) \log|x|$ + a harmonic function $H(x)$ in $\mathbf{R}^2$ + a bounded harmonic function $b(x)$.

(3) With the representation of u as in Theorem 1, we define the order of u as $\operatorname{ord} u = \operatorname{ord} s$. (For the definition of the order of a subharmonic function in $\mathbf{R}^n$, $n \geqslant 2$, we refer for instance to p. 143 of W. K. Hayman and P. B. Kennedy [2].) Note that ord u is independent of the representation. For, if $u = s_1 - \alpha_1 \log|x|$ is another representation of u outside a disc, then $\operatorname{ord} s = \operatorname{ord}(s + \alpha_1 \log|x|) = \operatorname{ord}(s_1 + \alpha \log|x|) = \operatorname{ord} s_1$.

3. Subharmonic extension in $\mathbf{R}^2$.

Terminology. In this section we denote by u a subharmonic function defined outside a compact set in $\mathbf{R}^2$. We say that a subharmonic function V in $\mathbf{R}^2$ extends u if $V = u$ outside a disc.

Now a part of Théorème 2 of M. Brelot [1] can be stated as follows:

Let u be a subharmonic function defined outside a disc in $\mathbf{R}^2$. Then $V = \lim D_r u$ exists locally uniformly, and $V \equiv \infty$ if and only if the flux at infinity of u is > 0.

As a consequence, for sufficiently large r, $D_r u$ extended by u is a (nonharmonic) subharmonic function in $\mathbf{R}^2$ if and only if flux $u > 0$. Essentially the same result is obtained below as a simple application of Theroem 1.

THEOREM 2. *Given u there exists a (nonharmonic) subharmonic function in* $\mathbf{R}^2$ *extending u if and only if* $\mu(u) > 0$.

PROOF. If there is a function q (nonharmonic) subharmonic in $\mathbf{R}^2$ such that $q = u$ outside a disc, then clearly $\mu(u) = \lim(M(r, q)/\log r) > 0$ since q is not harmonic.

Conversely, let $u(x) = s(x) - \alpha \log|x|$ be a representation of u. Let λ be the measure associated with s. Since $\mu(u) > 0$, $\|\lambda\| > \alpha$. Choose a compact K such that $\lambda(K) > \alpha$.

Write $s = s_1 + s_2$ where s_2 is the lograithmic potential corresponding to λ restricted to K and s_1 is a subharmonic function in $\mathbf{R}^2$ with associated measure λ restricted to $\mathbf{R}^2 - K$.

Since $s_2(x) - \lambda(K) \log|x| \to 0$ as $|x| \to \infty$, $D_r(s_2 - \alpha \log|x|) \to \infty$ locally uniformly as $r \to \infty$. Consequently $D_r u \to \infty$ locally uniformly.

Hence, for large r, $D_r u \geqslant u$ on $|x| = R$ which implies that $D_r u \geqslant u$ in $R \leqslant |x| < r$. Define the function q as $D_r u$ in $|x| < r$ extended by u in $|x| \geqslant r$. Then q is a subharmonic function in $\mathbf{R}^2$ extending u.

COROLLARY 1. *Let u be lower bounded but not bounded in* $|x| \geqslant r$. *Then there exists a (nonharmonic) subharmonic function in* $\mathbf{R}^2$ *extending u.*

PROOF. Let $u \geqslant m$ outside a disc. Then $\mu(u) \geqslant 0$. We show now that $\mu(u) > 0$ and hence the corollary follows from the above theorem.

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For that, suppose $\mu(u) = 0$. Then if h is the least harmonic majorant of u outside a disc, h is of the form $h(x) =$ a harmonic function $H(x)$ in $\mathbf{R}^2$ + a bounded harmonic function $b(x)$. (See Remark 2 above.)

This implies that $m \leqslant u(x) \leqslant H(x) + b(x)$ outside a disc and hence H is a constant, which in turn implies that u is bounded, a contradiction.

COROLLARY 2. *Let u be of finite nonintegral order. Then there exists a (nonharmonic) subharmonic function in $\mathbf{R}^2$ extending u.*

PROOF. In this case we show that $\mu(u) = \infty$ and hence the corollary follows from Theorem 2.

For this, suppose that $\mu(u) < \infty$. Then the least harmonic majorant of u outside a disc is of the form $\mu(u)\log|x|$ + a harmonic $H(x)$ in $\mathbf{R}^2$ + a bounded harmonic function $b(x)$.

Let $u(x) = s(x) - \alpha\log|x|$ be a decomposition of u. Then from $s - \alpha\log|x| \leqslant \mu(u)\log|x| + H + b$ outside a disc, it follows that ord s = ord H.

Since ord u (= ord s) is finite and H is harmonic in $\mathbf{R}^2$, it now follows (for instance from Theorem 2.1.5 of W. K. Hayman and P. B. Kennedy [2]) that ord u is an integer, a contradiction.

4. In higher dimensions. We state here two theorems in $\mathbf{R}^n$, $n \geqslant 3$, analogous to those proved earlier in $\mathbf{R}^2$.

THEOREM 1′. *Let u be a subharmonic function defined in $|x| > R$ in $\mathbf{R}^n$, $n \geqslant 3$. Let $r > R$. Then there exist a nonconstant subharmonic function $s(x)$ in $\mathbf{R}^n$ and a constant $\alpha \leqslant 0$ such that $u(x) = s(x) - \alpha|x|^{2-n}$ in $|x| > r$.*

THEOREM 2′. *Let u be a subharmonic function defined in $|x| > R$ in $\mathbf{R}^n$, $n \geqslant 3$, with associated measure μ. Let $r > R$. Then the following are equivalent:*

(i) $\lim M(r, u) = \infty$.

(ii) $\int_r^\infty |y|^{2-n}\,d\mu(y)$ *is divergent.*

(iii) *There exists a subharmonic function v in $\mathbf{R}^n$, not majorized by any harmonic function, extending u.*

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