

DERIVATIONS ON COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let A be a commutative Banach algebra with radical R and D be a derivation on A . If $K = \{x \in R: \text{for every } n \geq 1, D^n x \in R\}$, then $DA \subseteq R$ if and only if K is closed.

1. Introduction. In [10] Singer and Wermer showed that the range of a continuous derivation on a commutative Banach algebra is contained in the radical. They conjectured that the assumption of continuity is unnecessary.

In [5] Johnson proved the following result.

THEOREM. *If A is a commutative Banach algebra with identity e and if $D: A \rightarrow A$ is a derivation, then there exist orthogonal idempotents $e_0 \dots e_m$ in A with sum e such that $D(e_0 A)$ is contained in the radical of $e_0 A$ and such that each algebra $e_i A \dots e_m A$ has just one maximal ideal. If A is semisimple, then D is continuous.*

Let \mathbb{C} be the complex field and \mathbb{N} the set of all nonnegative integers.

Throughout this paper we suppose that A is a commutative Banach algebra. R and ΔA will denote, respectively, the radical and the spectrum of A .

If S is a linear operator from a Banach space X into a Banach space Y , then the separating space $\mathfrak{S}(S)$ of S is defined by

$$\mathfrak{S}(S) = \{y \in Y: \text{there are } x_n \rightarrow 0 \text{ in } X \text{ with } Sx_n \rightarrow y \text{ in } Y\}.$$

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2. Main results.

LEMMA 1. *Let $D: A \rightarrow A$ be a derivation and let*

$$K = \{x \in R: \text{for every } n \geq 1, D^n x \in R\}.$$

Then K is a prime ideal of A .

PROOF. For convenience take $D^0 x = x$ ($x \in A$). It is plain that K is an ideal of A . Let $a_1 a_2 \in K$ for some $a_1, a_2 \in A$ and let $a_1 \notin K$. We must show that $a_2 \in K$. Since $a_1 \notin K$ there is an $n \geq 0$ such that, for each $m < n$, $D^m a_1 \in R$ but $D^n a_1 \notin R$. Now by induction on i we prove that, for each $i \geq 0$, $D^i a_2 \in R$.

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For $i = 0$: Since

$$D^n(a_1a_2) = a_2(D^n a_1) + \sum_{j=1}^n \binom{n}{j} D^j(a_2)D^{n-j}a_1$$

and $a_1a_2 \in K$, then $a_2D^n a_1 \in R$.

On the other hand R is a primitive ideal so R is a prime ideal. Therefore by the fact that $D^n a_1 \notin R$ we have $a_2 \in R$.

Let $D^0 a_2 \cdots D^{i-1} a_2 \in R$. We proceed for i . By Leibnitz's formula

$$\begin{aligned} D^{n+1}(a_1a_2) &= \sum_{j=0}^{i-1} \binom{n+i}{j} D^j(a_2)D^{n+i-j}(a_1) + \binom{n+i}{i} D^i(a_2)D^n(a_1) \\ &\quad + \sum_{j=i+1}^{n+i} \binom{n+i}{j} D^j(a_2)D^{n-j+i}(a_1). \end{aligned}$$

Since $a_1a_2 \in K$, then $D^{n+i}(a_1a_2) \in R$. Now by the assumption and induction hypothesis we get $\binom{n+i}{i} D^i(a_2)D^n a_1 \in R$. As before this implies that $D^i(a_2) \in R$. Hence we have the result.

LEMMA 2. *Let A have a unique maximal ideal which is its radical R . If D is a derivation on A , $K = \{x \in R: \text{for every } n \geq 1, D^n x \in R\}$ and $\mathfrak{S}(D) \cap R \subseteq K$, then $DA \subseteq R$.*

PROOF. Suppose that $\mathfrak{S}(D) \cap R \subseteq K$. Then $\mathfrak{S}(D)$ is an ideal in A so $\mathfrak{S}(D) = A$ or $\mathfrak{S}(D) \subseteq R$ because R is the unique maximal ideal of A . In the first case $R = \mathfrak{S}(D) \cap R \subseteq K \subseteq R$ so $K = R$. In the second case $\mathfrak{S}(D) = \mathfrak{S}(D) \cap R \subseteq K$.

The operator D_1 from A into A/\bar{K} defined by $D_1(a) = D(a) + \bar{K}$ ($a \in A$), where \bar{K} denotes the closure of K , is continuous, by [9, Lemma 1.4]. Since $D(K) \subseteq K$, it follows that $D_1(K) = 0$ so $D_1(\bar{K}) = 0$.

Thus D defines a continuous derivation D_0 from A/\bar{K} into A/\bar{K} by

$$D_0(x + \bar{K}) = Dx + \bar{K} \quad \text{for all } x \in A.$$

Singer and Wermer's Theorem [2, Theorem 18.16] implies that $D_0(A/\bar{K}) \subseteq R/\bar{K}$ because R/\bar{K} is clearly the unique maximal ideal of A/\bar{K} . Since $\bar{K} \subseteq R$, then $DA \subseteq R$.

THEOREM 1. *If D is a derivation on A and $K = \{x \in R: \text{for every } m \geq 1, D^m x \in R\}$, then $DA \subseteq R$ if and only if $\mathfrak{S}(D) \cap R \subseteq K$.*

PROOF. Since a derivation on a Banach algebra maps its identity to zero then we can, and will, suppose that A has an identity, say 1.

If $DA \subseteq R$, then $K = R$ so $\mathfrak{S}(D) \cap R \subseteq R = K$.

Conversely suppose that $\mathfrak{S}(D) \cap R \subseteq K$. By Johnson's Theorem there exist orthogonal idempotents e_0, e_1, \dots, e_m in A with sum 1 such that $D(e_0A)$ is contained in the radical of e_0A and such that each algebra e_1A, \dots, e_mA has a unique maximal ideal.

Let $1 < i \leq m$ and take $K_i = \{x \in \text{Rad}(Ae_i): \text{for all } n, D^n x \in \text{Rad}(e_iA)\}$. Hence $K_i = e_iK$ and so $\mathfrak{S}(D) \cap \text{Rad}(e_iA) \subseteq \text{Rad}(e_iA)$. Therefore by the above lemma $D(e_iA) \subseteq \text{Rad}(e_iA)$. Since i was arbitrary, then $DA \subseteq R$.

The following definition is due to Cusack [3].

DEFINITION. A closed ideal I of a Banach algebra B is called a separating ideal if, for every sequence (a_n) in B , there is an $N > 1$ such that $(Ia_1, \dots, a_n)^- = (Ia_1 \dots a_n)^- (n > N)$.

Let R be the ideal spanned by n -fold products of elements of R .

LEMMA 3. *If $\bigcap_{n>1} R^n = \{0\}$ and D is a derivation on A , then $\mathfrak{S}(D)$ is nilpotent and $DA \subseteq R$.*

PROOF. Since $\mathfrak{S}(D) \cap R \subseteq R$, $\mathfrak{S}(D) \cap R$ is a radical commutative Banach algebra. Let $x \in \mathfrak{S}(D) \cap R$. Since $\mathfrak{S}(D)$ is a separating ideal by [6, Lemma 1], there is a positive integer N such that

$$(G(D)x^N)^- = (G(D)x^n)^- \quad \text{for all } n > N.$$

Hence by the Mittag-Leffler Theorem [4, Theorem 5.3]

$$(\mathfrak{S}(D)x^N)^- = \bigcap_{n>1} (\mathfrak{S}(D)x^n)^- = \bigcap_{n>1} (\mathfrak{S}(D)x^n)^- = \{0\}.$$

Therefore $x^{N+1} = 0$ and x is nilpotent. Since x was arbitrary $\mathfrak{S}(D) \cap R$ is a nilpotent ideal. Since K is a prime ideal of A , then $\mathfrak{S}(D) \cap R \subseteq K$.

Now by Theorem 1 this implies that $DA \subseteq R$ and then $\mathfrak{S}(D) \cap R = \mathfrak{S}(D)$ since $\mathfrak{S}(D) \subseteq \overline{DA} \subseteq R$.

COROLLARY 1. *If $I = \bigcap_{n>1} R^n$ is closed, then for every derivation D on A , $DA \subseteq R$.*

PROOF. Since for all n , R^n is an ideal of A then I is an ideal of A . If D is a derivation on A then $DI \subseteq I$ because $D(R^n) \subseteq R^{n-1}$ for $n > 2$.

Thus D gives a derivation D_1 on A/I defined by

$$D_1(x + I) = Dx + I \quad (x \in A).$$

It is clear that $\text{Rad}(A/I) = R/I$.

Now we prove that $\bigcap_{n>1} (R/I)^n = \{0\}$. Let $\xi \in \bigcap_{n>1} (R/I)^n$; then for each n , there is an element $\alpha_n \in R^n$ such that $\xi = \alpha_n + I$. Thus $\alpha_1 - \alpha_n \in I$ for all n , $I \subseteq R^n$, so $\alpha_1 \in R^n$ for all n , i.e. $\alpha_1 \in I$ and so $\xi = 0$.

Then, by Lemma 3, $D_1(A/I) \subseteq R/I$ which implies $DA \subseteq R$, and we have the result.

In [7] Loy proved that every derivation on a Banach algebra of formal power series is continuous. Since for a Banach algebra of formal power series $\bigcap_{n>1} R^n = \{0\}$ and R is an integral domain, the following result is a generalization of his result.

COROLLARY 2. *If $\bigcap_{n>1} R^n = \{0\}$ and R is an integral domain, then every derivation D on A is continuous.*

PROOF. If D is a derivation on A , then $\mathfrak{S}(D)$ is nilpotent. On the other hand R is an integral domain, so $\mathfrak{S}(D) = \{0\}$ and D is continuous.

THEOREM 2. *If D is a derivation on A and $K = \{x \in R: \text{for every } n > 1, D^n x \in R\}$, then $DA \subseteq R$ if and only if K is closed.*

PROOF. If $DA \subseteq R$, then $K = R$ is closed.

Conversely if K is closed, then since $DK \subseteq K$, D gives a derivation D_1 on A/K defined by $D_1(x + K) = Dx + K$ ($x \in A$).

Since $K \subseteq R$, then $\text{Rad}(A/K) = R/K$.

We prove $\bigcap_{n>1} (R/K)^n = \{0\}$. Let $x \in \bigcap_{n>1} (R/K)^n$. Then there are elements $\alpha_n \in R^n$ such that $x = \alpha_n + K$. Now for every $n > 1$, $\alpha_1 - \alpha_{n+1} \in K$; therefore $D^n(\alpha_1 - \alpha_{n+1}) \in K$.

On the other hand $D^n \alpha_{n+1} \in R$ so $D^n \alpha_1 \in R$. Since n was arbitrary, then $\alpha_1 \in K$ and $x = 0$. But x was arbitrary, so $\bigcap_{n>1} (R/K)^n = \{0\}$, and by Lemma 3, $DA + K \subseteq R$. Then $DA \subseteq R$ as we wanted.

COROLLARY. *Let every prime ideal of A be closed. Then for every derivation D on A , $DA \subseteq R$.*

PROOF. Take $K = \{x \in R: \text{for every } n > 1, D^n x \in R\}$. Since K is a prime ideal of A , then K is closed and, by Theorem 2, we have the result.

EXAMPLE. Let l^2 be the well-known Hilbert space. Define $T: l^2 \rightarrow l^2$ by

$$Tx = (0, \alpha_1 x_1, \alpha_2 x_2, \dots) \text{ for } x = (x_1, x_2, \dots),$$

where (α_n) are nonzero elements of \mathbb{C} such that $\alpha_n \rightarrow 0$.

Then T is quasinilpotent and $\bigcap_{n>1} T^n(l^2) = (0)$. Let B be the norm-closed subalgebra of $L(l^2)$ generated by $\{I, T\}$ where $I: l^2 \rightarrow l^2$ is the identity map. Now we prove that $\text{Rad } B = \overline{BT}$.

Since T is quasinilpotent, then $\overline{BT} \subseteq \text{Rad } B$.

On the other hand, let $x \in \text{Rad } B$, so there are polynomials $P_n(T)$ such that $P_n(T) \rightarrow x$. We can write $P_n(T) = B_n I + q_n(T)$, where $q_n(T) \subseteq BT$.

Let φ be a multiplicative linear functional on B , so $\varphi(p_n(T)) = B_n \rightarrow \varphi(x) = 0$. Therefore $q_n(T) \rightarrow x$. But $q_n(T) \subseteq BT$ and so $x \subseteq \overline{BT}$. Moreover $\bigcap_{n>1} (\text{Rad } B)^n = \{0\}$ and $B = \mathbb{C} \cdot I \oplus \text{Rad } B$. Hence for every derivation D on B , by Lemma 3, $DB \subseteq \text{Rad } B$. From Theorem 1, we get the following result.

THEOREM 1'. *If D is a derivation on A , then $DA \subseteq R$ if and only if for every $n > 1$ and $\varphi \in \Delta A$, $\varphi \circ D^n$ is continuous.*

PROOF. If $DA \subseteq R$, then for every $n > 1$ and $\varphi \in \Delta A$, $\varphi \circ D^n = 0$ which is continuous.

Conversely, for all $n > 1$ and $\varphi \in \Delta A$, let $\varphi \circ D^n$ be continuous. Let $x \in \mathfrak{C}(D) \cap R$, $n > 1$, and $\varphi \in \Delta A$. By the fact that $\varphi \circ D^n$ and $\varphi \circ D^{n+1}$ are continuous, we have $\varphi \circ D^n(x) = 0$. Since φ and n were arbitrary and $x \in R$, we conclude that $x \in K$ and by Theorem 1 we have the result.

For generalizing Singer and Wermer's Theorem first we prove the following lemma.

LEMMA 4. *Let A have a unique maximal ideal which is its radical. If D is a derivation on A and, for some n , D^n is continuous, then $DA \subseteq R$.*

PROOF. Let $\Delta A = \{\varphi\}$. Since D^n is continuous, for each $a \in A$ and $m \in \mathbb{N}$

$$\|D^m(a)\| \leq \max_{0 \leq j < n} \|D^j(a)\| \cdot \|D^n\|^i$$

where $m = in + r$, $i \in \mathbb{N}$, $0 \leq r < n$. So $\exp D$ is well defined and, for every $z \in \mathbb{C}$, the map $a \mapsto (\exp(zD)a)$ ($a \in A$) is a multiplicative linear functional. Therefore

$$\varphi(\exp(zD)a) = \varphi(a) \quad (a \in A).$$

Since $\varphi(a)$ is independent of z , this gives $\varphi \circ D = 0$, i.e. $DA \subseteq R$.

THEOREM 3. *If D is a derivation on A and, for some n , D^n is continuous, then $DA \subseteq R$.*

PROOF. As we state in the proof of Theorem 1 we can, and will, suppose that A has an identity, say 1. By Johnson's Theorem, there exist orthogonal idempotents e_0, e_1, \dots, e_m in A with sum 1 such that $D(e_0A)$ is contained in the radical of e_0A and such that each algebra e_1A, \dots, e_mA has a unique maximal ideal. Let $1 \leq i \leq m$. Since D^n is continuous, the restriction of D^n to e_iA is continuous and by the above lemma $D(e_iA) \subseteq \text{Rad}(e_iA)$ since i was arbitrary, then $DA \subseteq R$.

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