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REDUCTION THEOREMS FOR A CLASS OF SEMILINEAR EQUATIONS AT RESONANCE

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ABSTRACT. In solving equations of the form $Lu - Nu = p$ in a Hilbert space, where L is linear and N is nonlinear, the alternative method can sometimes be used to reduce the problem to one in a subspace. In this note previous reduction results are extended and at the same time the proofs are simplified. The approach is to use simple fixed point theorems in place of the traditional variational methods which are often quite delicate.

1. Introduction. This note is to extend previous results by Castro [6], Bates and Castro [3], Amann [1], Bates [2] and Mawhin [7, 8], while giving simpler proofs of the results in [1, 2, 3]. An application of the abstract results shows that under certain assumptions on g, the equation

$$y'' + g(t + y) = c = \text{constant}$$

has no 2π-periodic solutions if $c \neq 0$ and if $c = 0$ has a continuum of such solutions. This example was chosen because it seemed interesting; it is not a general representative of the class of equations treated here. In fact the abstract results below may be applied to semilinear elliptic and hyperbolic PDE's.

To proceed, let H be a (real or complex) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, let L be a linear operator in H, N be a nonlinear operator on H and let $p \in H$. Consider the question of solvability of

$$Lu - Nu = p. \tag{1.1}$$

In [7] Mawhin has

THEOREM A. *Suppose L is selfadjoint with spectrum σ and N has a selfadjoint Gâteaux derivative $N'(u)$ which satisfies*

(1.2) *There exist numbers $a \leqslant b$ with $[a, b] \cap \sigma = \varnothing$ and $aI \leqslant N'(u) \leqslant bI$ for each $u \in H$.*

Then (1.1) *is uniquely solvable with the solution depending (Lipschitz) continuously on p.*

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The main theorem of this paper will contain as a special case

THEOREM 1. *Suppose L and N are as above except* (1.2) *is replaced by*:

(1.2)′ *For each* $u \in H$ *there exist numbers* $a(u) \leq b(u)$ *with* $[a(u), b(u)] \cap \sigma = \varnothing$ *and such that* $a(u)I \leq N'(u) \leq b(u)I$.

(1.2)″ $-\infty < a \equiv \inf\{a(u): u \in H\} \leq \sup\{b(u): u \in H\} \equiv b < \infty$ *and for some constants* $C > 0$, $d < (b - a)/2$,

$$\|N(u) - (a + b)u/2\| \leq d\|u\| + C.$$

Then (1.1) *has a unique solution.*

REMARK. The value of Theorem 1 over Theorem A is that $[a, b] \cap \sigma \neq \varnothing$ is allowed; that is, the union of the numerical ranges of the operators $N'(u)$, $u \in H$, may have points of σ in its closure.

Whereas the above theorems are existence results we are mainly concerned here with reduction theorems associated with applying the alternative method to (1.1). To be specific, suppose H has the orthogonal decomposition $H = H_1 \oplus H_2$, with L leaving H_i invariant. Let P_i be the orthogonal projection onto H_i; then (1.1) is equivalent to the system

$$Lu_1 - P_1N(u_1 + u_2) = p_1, \tag{1.3a}$$

$$Lu_2 - P_2N(u_1 + u_2) = p_2, \tag{1.3b}$$

where for $z \in H$, $z_i = P_iz \in H_i$. Under certain hypotheses on L and N it may be possible to solve (1.3b) once u_1 is fixed, in which case solvability of (1.1) is reduced to solvability of (1.3a) with $u_2 = u_2(u_1)$ being the solution of (1.3b). Thus, we are concerned with imposing conditions on L and N so that (1.3b) has, for fixed u_1, a unique solution $u_2(u_1)$ which depends continuously on u_1. Theorems B and C below are along these lines and are extended in Theorem 2. Suppose

(1.4) L is selfadjoint with spectrum $\sigma \subset (-\infty, a - \varepsilon] \cup [a, b] \cup [b + \varepsilon, \infty)$, where $a \leq b$ and $\varepsilon > 0$;

(1.5) N is a continuous gradient operator such that $a\|u - v\|^2 \leq (Nu - Nv, u - v) \leq b\|u - v\|^2$ for $u, v \in H$.

Now suppose $\{E_\lambda\}$ is the resolution of the identity associated with L and let $P_1 = \int_a^b dE_\lambda$, $P_2 = I - P_1$ and $H_i = P_iH$, $i = 1, 2$.

THEOREM B (AMANN [1]). *Suppose* (1.4) *and* (1.5) *hold. Then for each* $u_1 \in H_1$, (1.3b) *has a unique solution* $u_2(u_1) \in H_2$. *Furthermore,* u_2 *depends continuously upon* u_1.

REMARK. In [1] Amann gave a variational proof of this result and in [8] Mawhin has since given a much simpler proof. It was Mawhin's proof which inspired this note.

A different theorem founded in [6] and improved in [3] and [2] involves weakening the hypotheses on N and strengthening those on L. Suppose

(1.6) The restriction of L to H_2 has compact resolvent,

(1.7) N is a continuous gradient operator such that, for $u \neq v$, $(a - \varepsilon)\|u - v\|^2 < (Nu - Nv, u - v) < (b + \varepsilon)\|u - v\|^2$, and

(1.8) $\limsup_{\|u\|\to\infty}\|Nu - (a+b)u/2\|/\|u\| < (b-a)/2 + \varepsilon$.

THEOREM C. *Suppose L satisfies* (1.4) *and* (1.6) *and that N satisfies* (1.7) *and* (1.8). *Then the conclusion of Theorem* B *holds.*

REMARK. Theorem C may be used to show that $y'' + \sin y = p(t)$ has 2π-periodic solutions for certain functions $p(t)$ where Theorem B fails to be of use.

Now suppose (perhaps after shifting L and N so that $a = -b$ in (1.4)):

(1.9) H has the orthogonal decomposition $H_1 \oplus H_2$ with H_i invariant under the closed operator L, and such that $\|(L|_{H_2})^{-1}\| \leq 1/l$ for some $l > 0$;

(1.10) For $u \neq v$, $\|Nu - Nv\| < l\|u - v\|$.

(1.11) For some constants $\bar{l} < l$ and $C > 0$, $\|Nu\| \leq \bar{l}\|u\| + C$.

Let P_i be the orthogonal projection onto H_i.

THEOREM 2. *Suppose that L and N satisfy* (1.9)–(1.11). *Then for each* $u_1 \in H_1$ *there exists a unique* $u_2(u_1) \in H_2$ *satisfying* (1.3b).

Lacking in this theorem is the continuity of $u_2(\cdot)$; however, suppose

(1.12) H_2 has the orthogonal decomposition $H_3 \oplus H_4$ with H_i invariant under L ($i = 3, 4$), H_3 finite dimensional and with $\|(L|_{H_4})^{-1}\| < 1/l$.

We have

THEOREM 3. *If* (1.9)–(1.12) *hold, then the function* $u_2(\cdot)$ *given by Theorem* 2 *is continuous.*

REMARKS. 1. Although it was not stated, Theorems B and C required H to be real; Theorems 2 and 3 do not.

2. If L is self-adjoint (or normal) satisfying (1.4) (respectively, $\sigma \subset \bar{D}((a+b)/2, (b-a)/2) \cup D^c((a+b)/2, (b-a)/2 + \varepsilon)$, where $D(z, r)$ is the open disc in $\mathbf{C}$ centered at z of radius r) then $L - (a+b)I/2$ satisfies (1.9) with $l = (b-a)/2 + \varepsilon$ and H_i chosen in the obvious way.

3. Condition (1.7) implies that $N - (a+b)I/2$ satisfies (1.10) (see **[4]** or **[8]**) and (1.11) follows from (1.8).

4. There is no compactness assumption on the resolvent of L and N need not be a gradient operator.

2. Proofs. Theorem 2 has a particularly simple proof using the following result of F. Browder **[5]**:

THEOREM D. *Let* $B = \{u\colon \|u\| \leq R\}$ *and* $S = \partial B$. *Let* $T\colon B \to H$ *satisfy* $\|Tu - Tv\| \leq \|u - v\|$ *for all* $u, v \in B$ *and* $T(S) \subset B$. *Then T has a fixed point in B.*

PROOF OF THEOREM 2. For fixed $u_1 \in H_1$, rewrite (1.3b) as

$$u_2 = L^{-1}(P_2N(u_1 + u_2) + p_2) \equiv Tu_2.$$

Note that

$$\|Tu_2 - T\bar{u}_2\| \leq \|(L|_{H_2})^{-1}\| \cdot \|N(u_1 + u_2) - N(u_1 + \bar{u}_2)\| < \|u_2 - \bar{u}_2\|$$

for $u_2 \neq \bar{u}_2$, by (1.9) and (1.10). Thus, T is nonexpansive and any fixed point is unique. Finally, (1.11) implies

$$\|Tu_2\|/\|u_2\| \leqslant \bar{l}/l + \left(\bar{l}\|u_1\| + C + \|p_2\|\right)/\left(l\|u_2\|\right). \tag{2.1}$$

Since u_1 is fixed, for $\|u_2\| = R$ sufficiently large, $\|Tu_2\| < \|u_2\|$. Theorem D completes the proof.

Proof of Theorem 1. Condition (1.2)′ implies that for each $u \in H$ there are points $c(u)$, $d(u) \in \sigma \cup \{a, b\}$ such that $(c(u), d(u)) \cap \sigma = \varnothing$ and $c(u) < a(u) \leqslant b(u) < d(u)$. Actually, c and d do not depend upon u. This follows from an extension of the "Intermediate Value Theorem for Derivatives" as outlined below. Suppose $c(u_1) \leqslant (N'(u_1)v, v) \leqslant d(u_1)$ and $c(u_2) \leqslant (N'(u_2)v, v) \leqslant d(u_2)$ for all $v \in H$ and suppose $d(u_1) \leqslant c(u_2)$. Then $[d(u_1), c(u_2)]$ is contained in the set $\{(N'(u)v, v)\colon u \in H\}$, where $v = u_2 - u_1$. To see this, consider the continuous functions $f, g\colon (0, 1] \to \mathbf{R}$ defined by

$$f(t) = (N(u_2 + t(u_1 - u_2)) - N(u_2), u_1 - u_2)/t,$$
$$g(t) = (N(u_1 + t(u_2 - u_1)) - N(u_1), u_2 - u_1)/t.$$

By the Mean Value Theorem (see **[9]**) we may write

$$f(t) = (N'(z(t))(u_1 - u_2), u_1 - u_2), \quad g(t) = (N'(y(t))(u_2 - u_1), u_2 - u_1) \tag{2.2}$$

for some points $z(t), y(t)$ lying on the line segment joining u_1 and u_2. Now, f and g are continuous at 0, and defined by $f(0) = (N'(u_2)(u_1 - u_2), u_1 - u_2)$, $g(0) = (N'(u_1)(u_2 - u_1), u_2 - u_1)$. Note also that $f(1) = g(1)$, so that $f([0, 1]) \cup g([0, 1])$ is connected, and by the representation (2.2), $[g(0), f(0)]$ is contained in

$$\{(N'(u)(u_2 - u_1), u_2 - u_1)\colon u \text{ lies on the line segment joining } u_1 \text{ and } u_2\}.$$

For the sake of simplicity suppose $a, b \in \sigma$. In Theorem 2 take $l = (b - a)/2$, $\hat{L} \equiv L - (b + a)I/2$ in place of L, $\hat{N} \equiv N - (b + a)I/2$ in place of N, $H_1 = \{0\}$, $H_2 = H$, $\bar{l} = d$. It is easy to see that (1.9) and (1.11) hold. By the Mean Value Theorem $\|\hat{N}u - \hat{N}v\| \leqslant \|\hat{N}'(z)\| \cdot \|u - v\|$ for some z on the line segment joining u and v. Now $\hat{N}'(z)$ is selfadjoint and $a - (a + b)/2 < a(z) - (a + b)/2 \leqslant (\hat{N}'(z)v, v) \leqslant b(z) - (a + b)/2 < b - (a + b)/2$, so $\|\hat{N}'(z)\| < (b - a)/2$ and (1.10) holds. This completes the proof.

Proof of Theorem 3. Write (1.3b) as

$$Lu_3 - P_3N(u_1 + u_3 + u_4) = p_3, \tag{2.3}$$
$$Lu_4 - P_4N(u_1 + u_3 + u_4) = p_4, \tag{2.4}$$

where $u_2 = u_3 + u_4$, $p_2 = p_3 + p_4 \in H_3 \oplus H_4$, and P_i is the orthogonal projection onto H_i, $i = 3, 4$. Fix $u_1 + u_3 \in H_1 \oplus H_3$ and consider (2.4) rewritten as $u_4 = (L|_{H_4})^{-1}(P_4N(u_1 + u_3 + u_4) + p_4) \equiv Ku_4$. From conditions (1.10) and (1.12) it is easy to see that K is a strict contraction, and has, by the Contraction Mapping Theorem, a unique fixed point $u_4(u_1 + u_3)$. Furthermore, $u_4\colon H_1 \oplus H_3 \to H_4$ is Lipschitz continuous. Now, from Theorem 2, we know that (2.3), (2.4) has a unique solution for fixed u_1. Write this as $u_2(u_1) = u_3(u_1) + \bar{u}_4(u_1)$. The uniqueness implies that $\bar{u}_4(u_1) = u_4(u_1 + u_3(u_1))$. Therefore, in order to prove that $u_2\colon H_1 \to H_2$ is continuous, it suffices to show that $u_3\colon H_1 \to H_3$ is continuous. From (2.1) it follows that $u_1 \to u_2(u_1)$ takes bounded sets into bounded sets and so the same is true of the mapping $u_3(\cdot)$. Suppose that $\{u_1^n\} \subset H_1$ converges to u_1. Then since $\dim H_3 < \infty$ we may assume that a subsequence has been taken so that $\{u_3(u_1^n)\}$ converges to a point $u_3 \in H_3$. Since N and $u_4(\cdot)$ are continuous and L is closed, it follows that u_3 and $u_4(u_1 + u_3)$ satisfy (2.3) and (2.4). The uniqueness of the solution implies $u_3 = u_3(u_1)$. It follows that u_3, and hence u_2, is continuous.

3. An example. Consider the problem

$$y'' + g(y + t) = c = \text{constant}, \tag{3.1}$$

$$y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0, \tag{3.2}$$

where g is a differentiable 2π-periodic function satisfying

$$|g'(x)| \leqslant 1 \quad \text{for all } x \in \mathbf{R}, \tag{3.3}$$

$$\{x\colon |g'(x)| = 1\} \text{ has measure zero}, \tag{3.4}$$

$$\int_0^{2\pi} g(s)\, ds = 0. \tag{3.5}$$

We will show that for $c \neq 0$ this problem has no solution while for $c = 0$ there is a continuum of solutions. Let $H = L_2(0, 2\pi)$, $H_1 = \mathbf{R}$, P_1 the projection defined by $P_1 f = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$, $P_2 = I - P_1$, $H_2 = P_2 H$. Let L be the closure of the operator L_0 defined by $\operatorname{dom} L_0 = \{y \in C^2[0, 2\pi]\colon y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0\}$, $L_0 y = -y''$. Then L is selfadjoint with spectrum $\sigma = \{k^2\colon k = 0, 1, \ldots\}$ and $\operatorname{Ker} L = H_1$. Let N be given by $N(y) = g(y + t)$. Then N is continuous on H and, with $l = 1$, satisfies (1.10), for if $y \neq z \in H$,

$$\begin{aligned}\|Ny - Nz\|^2 &= \int_0^{2\pi} (g(y(t) + t) - g(z(t) + t))^2\, dt \\ &= \int_0^{2\pi} \left[\int_0^1 g'(z(t) + t + s(y(t) - z(t)))\, ds\right]^2 (y(t) - z(t))^2\, dt.\end{aligned}$$

On a set of positive measure $y(t) \neq z(t)$ and hence, for such values of t, (3.4) implies $|g'(z(t) + t + s(y(t) - z(t)))| < 1$ a.e. for $s \in [0, 1]$. This gives (1.10). Using (3.3) and (3.5) it can be shown (integrate by parts) that $|g(x)| \leqslant \frac{1}{2}|x| + C$ for C sufficiently large; hence, (1.11) is valid. Clearly (1.9) holds and also (1.12) with $H_3 = \operatorname{span}\{\sin t, \cos t\}$ and $H_4 = \overline{\operatorname{span}}\{\sin kt, \cos kt\colon k = 2, 3, \ldots\}$. By Theorems 2 and 3, there exists a continuous function $u_2\colon \mathbf{R} \to H_2$ so that (3.1), (3.2) has a solution if and only if there exists a constant x such that

$$\bar{g}(x) \equiv \frac{1}{2\pi}\int_0^{2\pi} g(t + x + u_2(x)(t))\, dt = c. \tag{3.6}$$

We now show that $\bar{g} \equiv 0$, i.e., if $c \neq 0$, (3.1), (3.2) has no solution. This implies that if $c = 0$, $x + u_2(x)(t)$ solves (3.1), (3.2) for each $x \in \mathbf{R}$. In general, $u_2(x) \neq u_2(z)$ when $x \neq z$ so the continuum of solutions is not merely a translate of $\mathbf{R}$ in $H = \mathbf{R} + H_2$. Suppose y solves (3.1), (3.2). Multiplying (3.1) by $1 + y'(t)$ and integrating over $[0, t]$ gives

$$y'(t) + y'^2(t)/2 + G(t + y(t)) - cy(t) + D = ct.$$

where D is a constant and $G' = g$. Now divide by t and let t tend to infinity. Since y and y' are bounded (2π-periodic) and $G(t + y(t))/t \to 0$ by (3.5), we must conclude that $c = 0$. Actually, this shows that if $c \neq 0$ there are no solutions y of (3.1) for which y and y' are bounded.

I would like to thank Professor F. Odeh for pointing out that (3.1), (3.2) has no solutions in the case $c \neq 0$ with $g(x) \equiv \sin x$.

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