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CONVERGING FACTORS FOR CONTINUED FRACTIONS $K(a_n/1)$, $a_n \to 0$

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ABSTRACT. Converging factors for continued fractions $K(a_n/1)$ are used to enhance convergence either by accelerating the convergence process or by altering the region of convergence if the a_n's are functions of a complex variable. The first results concerning the use of converging factors to accelerate convergence in the important case $a_n \to 0$ are presented in this paper.

The approximants, A_n/B_n, of the continued fraction

$$\frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1} + \cdots, \tag{1}$$

where each a_n is a nonzero complex number, can be generated in the following way.

Let $t_n(z) = a_n/(1+z)$, $n \geqslant 1$, and $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$, $n \geqslant 2$. Then $A_n/B_n = T_n(0)$, $n \geqslant 1$. Let us assume that (1) converges, i.e. $\operatorname{Lim}_{n\to\infty} T_n(0) = T$ exists.

Complex numbers $\mu_1, \mu_2, \ldots$ are called *converging factors* of (1) provided

$$\operatorname*{Lim}_{n\to\infty} T_n(\mu_n) = \operatorname*{Lim}_{n\to\infty} T_n(0) = T.$$

For limit-periodic continued fractions (1) (i.e., $a_n \to a$ as $n \to \infty$), something is known about converging factors that accelerate convergence. See, e.g., [**2**, **3**, **6**, and **7**]. However, these investigations are restricted to the case $a \neq 0$. In this paper the case $a = 0$ is considered.

The following theorem [**2**] is a generalization of a theorem appearing in [**4**] and is basic to the study of converging factors of limit-periodic fractions. Set $a_n = \alpha_n(\alpha_n + 1)$, where $|\alpha_n| < |\alpha_n + 1|$, $n \geqslant 1$, and $\operatorname{Lim}_{n\to\infty} a_n = a = \alpha(\alpha+1)$, where $|\alpha| < |\alpha + 1|$. The imposed conditions established by the inequalities imply $a_n \neq -\frac{1}{4}$, and $a \neq -\frac{1}{4}$.

THEOREM 1.
$$\underline{\operatorname{Lim}}\, |\mu_n - (\alpha+1)| > 0 \Rightarrow \operatorname*{Lim}_{n\to\infty} T_n(\mu_n) = \operatorname*{Lim}_{n\to\infty} T_n(0).$$

In [**3**] the author developed a geometrical approach to the use of converging factors of the form $\mu_n \equiv \alpha$ for accelerating convergence of certain limit-periodic fractions. The elementary techniques involved gave fairly accurate truncation error estimates. Waadeland, in essence, employed $\mu_n \equiv \alpha$ in his study of limit-periodic T-fractions [**7**]. More recently, Thron and Waadeland [**6**] reported the far more general result that follows.

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THEOREM 2. *Let* $a_n \to a \neq 0$, $|\arg(a + \frac{1}{4})| < \pi$. *Assume that, for all* $n \geq 1$, $|a_n - a| < \min\{\frac{1}{2}(|a + \frac{1}{4}| + \frac{1}{4} - |a|), |a|/2\}$. *Set* $d_n = \max_{m \geq n}|a_m - a|$. *Then*

$$\left|\frac{T - T_n(\alpha)}{T - T_n(0)}\right| \leq 2d_n \frac{|a| + |\frac{1}{2} + a + \sqrt{\frac{1}{4} + a}\,|}{|a|(\frac{1}{4} + |\frac{1}{4} + a| - |a|)}, \qquad \mathrm{Re}\left(\sqrt{\tfrac{1}{4} + a}\,\right) > 0.$$

Set

$$T_k^{(n)} = \frac{a_{n+1}}{1} + \frac{a_{n+2}}{1} + \cdots + \frac{a_{n+k}}{1}, \qquad n \geq 0, k \geq 1,$$

and $T^{(n)} = \mathrm{Lim}_{k\to\infty} T_k^{(n)}$, $n \geq 0$, provided these limits exist. Let $T = T^{(0)}$. Then $T^{(n)}$ is the "tail end" of (1), and it is natural to use α as a constant converging factor, since $T^{(n)} \to \alpha$ as $n \to \infty$. See, e.g., [5, p. 286].

However, this very convenient factor fails to be of any value if $a = 0$ ($\alpha = 0$), for we then have merely the traditional approximants of (1). Under certain circumstances the converging factors $\mu_n = \alpha_{n+1}$, $n \geq 1$, accelerate convergence in this special case. The use of the α_n notation instead of the a_n notation facilitates progress in this direction, since the α_n's are the *attractive fixed points* of the t_n's [1, pp. 6–21] and the geometrical approach to the convergence behavior of (1) initiated by this concept has proven to be of value in the past [3].

A "reluctant" convergence of $\{\alpha_n\}$ to 0 sets the stage for the advantageous use of these converging factors. Here $\alpha_n = -\frac{1}{2} + \sqrt{\frac{1}{4} + a_n}$, $\mathrm{Re}\sqrt{\frac{1}{4} + a_n} > 0$.

THEOREM 3. *If* (i) $\max_{m \geq n}|\alpha_m - \alpha_{m+1}| \leq \varepsilon_n|\alpha_{n+1}|$, $n = 1, 2, \ldots,$ *where* $0 \leq \varepsilon_n \leq 1$, *and* (ii) $0 < |\alpha_m| < \sigma_n \leq \frac{1}{5}$, $m \geq n$, $n = 1, 2, \ldots,$ *are satisfied, then*

$$|T_n(\alpha_{n+1}) - T| < \frac{\sigma_n \varepsilon_n}{(1 - 5\sigma_n)^2} \cdot |T_n(0) - T|$$

where $\mathrm{Lim}_{n\to\infty} \sigma_n = 0$.

PROOF. Let $h_1 = 1$,

$$h_n = 1 + \frac{a_n}{1} + \frac{a_{n-1}}{1} + \cdots + \frac{a_2}{1}, \qquad n \geq 2.$$

The following equation is easily obtained (see [6]).

$$\left|\frac{T - T_n(\alpha_{n+1})}{T - T_n(0)}\right| = \left|\frac{T^{(n)} - \alpha_{n+1}}{T^{(n)}}\right| \cdot \left|\frac{h_n}{h_n + \alpha_{n+1}}\right|. \tag{2}$$

Let us first consider the expression $|T^{(n)} - \alpha_{n+1}|$ in (2). Set $\rho_{m-1} = T^{(m-1)} - \alpha_m$, $d_m = |\alpha_{m+1} + 1| - |\alpha_m|$, $f_m = |\alpha_m(\alpha_m - \alpha_{m+1})|$, $D_n = \min_{m \geq n} d_m$, and $F_n = \max_{m \geq n} f_m$, $m \geq 1, n \geq 1$. Then

$$|\rho_{m-1}| = \left|\frac{\alpha_m(\alpha_m + 1)}{1 + T^{(m)}} - \alpha_m\right| \leq \frac{|\alpha_m|(|\alpha_m - \alpha_{m+1}| + |\rho_m|)}{|1 + \alpha_{m+1}| - |\rho_m|}. \tag{3}$$

As in [6], we wish to find $R_n > 0$ such that $|\rho_m \leq R_n$ for $m \geq n$. Assuming $|\rho_m| \leq R_n$ in (3), we have

$$|\rho_{m-1}| \leq \frac{|\alpha_m|(|\alpha_m - \alpha_{m+1}| + R_n)}{|1 + \alpha_{m+1}| - R_n}.$$

The expression on the right is $< R_n$ provided $f_m < d_m R_n - R_n^2$. Since $f_m < F_n$ and $D_n < d_m$ for $m > n$, $f_m < d_m R_n - R_n^2$ if $F_n \leq D_n R_n - R_n^2$. This last inequality is satisfied if $R_n = F_n D_n/(D_n^2 - 2F_n)$, as can be routinely verified by showing that $|\alpha_n| < \frac{1}{5}$ implies $4F_n < D_n^2$.

Now, $\text{Lim}_{n\to\infty} T^{(n)} = 0$ and $\text{Lim}_{n\to\infty} \alpha_{n+1} = 0$ imply $\text{Lim}_{m\to\infty} \rho_m = 0$. Hence, there exists $k > 0$ for fixed m and n $(m > n)$ such that $|\rho_{m+k}| < R_n$. Then $|\rho_{m-1}| < R_n$; i.e., $|T^{(m-1)} - \alpha_m| < R_n$, $m > n$.

Turning now to the first factor in the right side of (2),

$$\left|\frac{T^{(n)} - \alpha_{n+1}}{T^{(n)}}\right| = \frac{1}{\left|\dfrac{\alpha_{n+1}}{T^{(n)} - \alpha_{n+1}} + 1\right|} < \frac{1}{\left|\dfrac{\alpha_{n+1}}{T^{(n)} - \alpha_{n+1}}\right| - 1},$$

we see that

$$\left|\frac{T^{(n)} - \alpha_{n+1}}{\alpha_{n+1}}\right| < \frac{R_n}{|\alpha_{n+1}|}$$

$$< \frac{\max_{m>n}|\alpha_m - \alpha_{m+1}|}{|\alpha_{n+1}|} \cdot \max_{m>n}|\alpha_m| \cdot \frac{D_n}{D_n^2 - 2F_n} < \varepsilon_n \sigma_n \cdot \frac{1}{1 - 4\sigma_n},$$

since $1 - 2\sigma_n < D_n < 1$ and $F_n < 2\sigma_n^2$. Therefore,

$$\left|\frac{T^{(n)} - \alpha_{n+1}}{T^{(n)}}\right| < \frac{\varepsilon_n \sigma_n}{1 - 5\sigma_n}. \tag{4}$$

Inverting the second factor in (2),

$$\left|\frac{h_n + \alpha_{n+1}}{h_n}\right| > 1 - \frac{|\alpha_{n+1}|}{|h_n|} > 1 - 2|\alpha_{n+1}| > 1 - 2\sigma_n, \tag{5}$$

since $|h_n| > \frac{1}{2}$ in the Worpitzy circle, $a_n \in \{z: |z| < \frac{1}{4}\}$ [**8**, p. 60], and condition (ii) implies $|a_n| < \frac{1}{4}$. Combining (4) and (5) gives the conclusion of Theorem 3.

EXAMPLE. Let $|\alpha_1| < 10^{-3}$ and $\alpha_n = (.52)^{n-1}\alpha_1$ for $n > 2$. Then $\varepsilon_n < 9.3 \times 10^{-1}$ and $\sigma_n = (.52)^{n-1} \times 10^{-3}$ for $n > 2$. Theorem 3 gives, e.g.,

$$|T_2(\alpha_3) - T| < 4.9 \times 10^{-4}|T_2(0) - T|$$

and

$$|T_{10}(\alpha_{11}) - T| < 2.6 \times 10^{-6}|T_{10}(0) - T|.$$

In general, if the α_n's are quite small, then an improvement on the order of magnitude of $|\alpha_1|$ occurs in the first calculation.

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