PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

RELATIVE WEAK CONVERGENCE IN SEMIFINITE VON NEUMANN ALGEBRAS

VICTOR KAFTAL

ABSTRACT. An operator is compact relative to a semifinite von Neumann algebra, i.e., belongs to the two-sided closed ideal generated by the finite projections relative to the algebra, if and only if it maps vector sequences converging weakly relative to the algebra into strongly converging ones (generalized Hilbert condition). The generalized Wolf condition characterizes the class of almost Fredholm operators.

Introduction. The elements of the two-sided closed ideal $\mathcal{J}$ generated by the projections finite relative to a von Neumann algebra $\mathcal{A}$ are called compact operators of $\mathcal{A}$ and it has been shown (see [7, 4]) that they satisfy many of the properties of the compact operators on a Hilbert space. The weak convergence of vectors plays an important role in classical operator theory. The aim of this paper is to study a relative weak (RW) convergence that could play an analogous role in the operator theory relative to a semifinite von Neumann algebra.

A bounded sequence of vectors x_n is defined to converge RW to x if $Px_n \underset{s}{\rightarrow} Px$ for every projection P finite relative to $\mathcal{A}$.

This convergence is shown to be not equivalent (apart from trivial cases) to weak or strong convergence. The classical Hilbert characterization is extended to the compact operators of $\mathcal{A}$ and a generalized Wolf property (see [8]) is used to characterize a new class $\tilde{\mathfrak{F}}_+$ called almost left-Fredholm [5]. We remark that the RW convergence can be used, in analogy with Calkin's construction (see [2]), to obtain a representation of the generalized Calkin algebra $\mathcal{A}/\mathcal{J}$.

Finally a RW topology is defined on the unit ball of the predual of $\mathcal{A}$ and both the generalized Hilbert and Wolf properties are reformulated in a space-free setting.

The author wishes to thank M. Sonis, L. Brown and P. Fillmore for valuable suggestions.

1. The relative weak convergence. Let H be a Hilbert space, let $L(H)$ be the algebra of all bounded linear operators on H, let $\mathcal{A}$ be a semifinite von Neumann algebra on H and $\mathcal{P}(\mathcal{A})$ be the set of the projections of $\mathcal{A}$. Let $\mathcal{J}$ be the ideal of compact operators (relative to $\mathcal{A}$), i.e., the norm closed two-sided ideal of $\mathcal{A}$ generated by the finite projections of $\mathcal{A}$. It is known that $\mathcal{J}$ is proper iff $\mathcal{A}$ is

---

Received by the editors April 28, 1980. Some of the results in this paper were presented to the American Mathematical Society at the 85th Annual Meeting at Biloxi, Mississippi, January 1979.

1980 *Mathematics Subject Classification*. Primary 46L10, 47C15; Secondary 47A53, 47B05.

Key words and phrases. Compact operator, almost Fredholm operator, relative weak convergence, Calkin algebra, Wolf Theorem, semifinite von Neumann algebra.

infinite and that it is the maximal two-sided ideal of $\mathcal{A}$ which does not contain infinite projections (see Breuer [1] and for further properties Kaftal [4] and Sonis [7]).

Since a projection is finite relative to the algebra $L(H)$ iff it is finite in the euclidean sense, i.e., iff it is a finite sum of one-dimensional projections, a sequence $x_n \in H$ converges weakly to x ($x_n \underset{w}{\to} x$) if and only if, for every projection P finite relative to $L(H)$, Px_n converges strongly to Px ($Px_n \underset{s}{\to} Px$). This suggests the following generalization:

Definition 1. We say that a sequence $x_n \in H$ converges to x weakly relative to $\mathcal{A}$ ($x_n \underset{\mathrm{RW}}{\to} x$) if $\|x_n\|$ is bounded and if, for every projection $P \in \mathcal{P}(\mathcal{A})$ finite relative to $\mathcal{A}$, $Px_n \underset{s}{\to} Px$.

Let us note that a weakly convergent sequence is necessarily bounded, but as the following example shows, there are unbounded sequences satisfying the second part of Definition 1.

Example 2. Let H and K be infinite-dimensional separable Hilbert spaces with orthonormal bases $\{e_n\}$, $\{f_n\}$ respectively and let $C(K)$ be the factor of the scalar multiples of the identity I_K on K. Then $\mathcal{A} = L(H) \otimes C(K)$ is an infinite von Neumann factor of type I and $x_n = \sum_{i=n}^{2n} e_i \otimes f_i$ is an unbounded sequence. Let P be a finite protection of $\mathcal{A}$. Then there is a finite projection P_0 in $L(H)$ such that $P = P_0 \otimes I_K$ and without loss of generality we may assume that it is one-dimensional, i.e., that $P_0 = (\cdot, e)e$ for a unit $e \in H$. Then $\|Px_n\|^2 = \sum_{i=n}^{2n} \|P_0 e_i\|^2 \|f_i\|^2 = \sum_{i=n}^{2n} |(e_i, e)|^2 \to 0$. Let us further use this setting to note that the sequences $e_n \otimes f$ and $e \otimes f_n$ are converging to zero, the first one RW but not strongly, the second weakly but not RW.

We shall analyze now the relations between the RW convergence and the other convergences in H. Our main tool shall be

Proposition 3. *Let $Q \in \mathcal{P}(\mathcal{A})$ be infinite. Then there is an orthonormal sequence $x_n \underset{RW}{\to} 0$ in QH.*

Proof. Since Q is infinite there is a $Q' = \sum_{n=0}^{\infty} Q_n < Q$ with $Q_n \in \mathcal{P}(\mathcal{A})$ and $Q_n \sim Q_0 \neq 0$ for every n. As $\mathcal{A}$ is semifinite, there is a finite $0 \neq R < Q_0$ in $\mathcal{P}(\mathcal{A})$. Let $0 \neq y_0 \in \mathrm{RH}$ and let $P_0 = E_{y_0}(\mathcal{A}') \in \mathcal{A}$ be the cyclic projection on y_0. Then $0 \neq P_0 < R < Q_0$ and P_0 too is finite. Let $P_n < Q_n$ be the image of P_0 under the equivalence $Q_0 \sim Q_n$, let $U_n \in \mathcal{A}$ be the partial isometry mapping P_0H onto P_nH and let $P' = \sum_{n=0}^{\infty} P_n < Q$. Since $\mathcal{A}_{P_0}$, which we shall identify with $P_0\mathcal{A}P_0 \subset \mathcal{A}$, is finite and has a separating vector y_0, then it has a nonzero trace vector x_0 (see Theorem 4 III 1, Proposition 1 I 6 and Theorem 1 I 4, Dixmier [3]). We can choose $\|x_0\| = 1$ and define $x_n = U_n x_0 \in P_nH$. Then x_n is an orthonormal sequence in $P'H \subset QH$. We are going to prove that $x_n \underset{\mathrm{RW}}{\to} 0$. Define $\phi = \sum_{n=0}^{\infty} \omega_{x_n}$ where $\omega_x(A) = (Ax, x)$. It is easy to see that ϕ is a semifinite normal trace on $(\mathcal{A}_{P'})^+$. Let $S \in \mathcal{P}(\mathcal{A})$ be finite. Since $\|Sx_n\|^2 = (P'SP'x_n, x_n)$, assume that $S < P'$. Apply Lemma 1 by Peligrad and Zsido' [6] to S, ϕ and $\mathcal{A}_{P'}$ and find a set of mutually orthogonal central projections E_γ of sum the identity of $\mathcal{A}_{P'}$ such that $\phi(SE_\gamma) < \infty$

for every $\gamma \in \Gamma$. As $1 = \|x_0\|^2 = \Sigma_{\gamma\in\Gamma}\|E_\gamma x_0\|^2$, for every $\varepsilon > 0$ we can find a finite index set $\Lambda \subset \Gamma$ such that $\Sigma_{\gamma\notin\Lambda}\|E_\gamma x_0\|^2 < \varepsilon/2$. Consider

$$\|Sx_n\|^2 = \sum_{\gamma\in\Gamma} \|E_\gamma Sx_n\|^2 = \sum_{\gamma\in\Lambda} \|SE_\gamma x_n\|^2 + \sum_{\gamma\notin\Lambda} \|SE_\gamma x_n\|^2.$$

Then

$$\begin{aligned}\sum_{\gamma\notin\Lambda} \|SE_\gamma x_n\|^2 \leqslant \sum_{\gamma\notin\Lambda} \|E_\gamma U_n x_0\|^2 &= \sum_{\gamma\notin\Lambda} \|U_n E_\gamma x_0\|^2 \\ &= \sum_{\gamma\notin\Lambda} \|E_\gamma x_0\|^2 < \varepsilon/2\end{aligned}$$

since the U_n are partial isometries and commute with the E_γ. On the other hand $\phi(SE_\gamma) = \Sigma_{n=0}^\infty \omega_{x_n}(SE_\gamma) = \Sigma_{n=0}^\infty\|SE_\gamma x_n\|^2 < \infty$; therefore $\|SE_\gamma x_n\| \to 0$ for every γ. Thus $\Sigma_{\gamma\in\Lambda}\|SE_\gamma x_n\|^2 < \varepsilon/2$ for $n > N$, which implies $\|Sx_n\|^2 < \varepsilon$ for every $n > N$, i.e., $Sx_n \underset{s}{\to} 0$. Q.E.D.

THEOREM 4. (a) *The strong convergence implies the RW convergence, which in turn implies the weak convergence.*

(b) *The strong convergence and the RW convergence coincide if and only if $\mathcal{Q}$ is finite.*

(c) *The weak convergence and the RW convergence coincide if and only if all the finite projections of $\mathcal{Q}$ have finite euclidean dimension.*

PROOF. The first implication in (a) is obvious. Let $x_n \underset{\mathrm{RW}}{\to} x$. Because of the semifiniteness of $\mathcal{Q}$, the identity $I \in \mathcal{Q}$ can be decomposed into a sum $I = \Sigma_{\gamma\in\Gamma} P_\gamma$ of mutually orthogonal finite projections (see Proposition 7, Corollary 1 III 2, Dixmier [3]). For every $\gamma \in \Gamma$ we have $P_\gamma x_n \underset{s}{\to} P_\gamma x$ and hence for every $y \in H$, we have $(x_n, P_\gamma y) \to (x, P_\gamma y)$. As the linear envelope of $\bigcup_{\gamma\in\Gamma} P_\gamma H$ is strongly dense in $H = \Sigma_{\gamma\in\Gamma} \oplus P_\gamma H$ and as $\|x_n\|$ is bounded by definition, we have $x_n \underset{w}{\to} x$. Since $I \in \mathcal{P}(\mathcal{Q})$ is finite iff $\mathcal{Q}$ is finite, (b) is an obvious corollary of Proposition 3.

(c) Let us assume that all the finite projections of $\mathcal{Q}$ have finite euclidean dimension. If $x_n \underset{w}{\to} x$ and $P \in \mathcal{P}(\mathcal{Q})$ is finite, then $\|x_n\|$ is bounded and $Px_n \underset{s}{\to} Px$; hence $x_n \underset{\mathrm{RW}}{\to} x$. Moreover, because of (a), if $x_n \underset{\mathrm{RW}}{\to} x$, then $x_n \underset{w}{\to} x$. On the other hand, if the weak and the RW convergences coincide, then every finite projection has finite euclidean dimension. Otherwise there would be in its range an infinite orthonormal sequence weakly but surely not RW converging to zero. Q.E.D.

REMARK 5. It is easy to see that a von Neumann algebra satisfies condition (c) iff it is the sum of a finite number of type I factors with finite commutant.

Because of (a) the RW limit is unique and it is also easy to see that subsequences of RW converging sequences RW converge to the same limit. All that however would be false if the algebra were not semifinite. In type III algebras, for instance, the second part of the definition would "disappear" and any bounded sequence would "converge" to every vector of H.

We can now extend to von Neumann algebras some well-known classical properties of the weak convergence and of the compact operators.

PROPOSITION 6. (a) *If* $K \in \mathcal{J}$ *and* $x_n \underset{RW}{\to} x$, *then* $Kx_n \underset{s}{\to} Kx$.

(b) *If* $A \in \mathcal{Q}$ *and* $x_n \underset{RW}{\to} x$, *then* $Ax_n \underset{RW}{\to} Ax$.

PROOF. (a) Without loss of generality we can assume that $x = 0$. K satisfies the generalized Rellich condition, i.e., for every $\eta > 0$ there is a $P \in \mathcal{P}(\mathcal{Q})$ such that $\|KP\| \leqslant \eta$ and $I - P$ is finite (see Theorem 1.3, **[4]**). Thus $(I-P)x_n \underset{s}{\to} 0$ and hence $K(I-P)x_n \underset{s}{\to} 0$. As $\|K - K(I-P)\| < \eta$ and $\|x_n\|$ is bounded by definition, we have $Kx_n \underset{s}{\to} 0$.

(b) If $P \in \mathcal{P}(\mathcal{Q})$ is finite, $PA \in \mathcal{J}$ and hence $P(Ax_n) \underset{s}{\to} P(Ax)$, i.e., $Ax_n \underset{\mathrm{RW}}{\to} Ax$. Q.E.D.

Part (a) of this proposition shows that the RW convergence could be equivalently defined in terms of compact operators instead of finite projections. The following theorem is a generalization of Hilbert's characterization of the compact operators.

THEOREM 7. *A is compact in* $\mathcal{Q}$ *iff it maps RW converging sequences into strongly converging ones.*

PROOF. Let $\tilde{\mathcal{J}}$ be the set of the operators of $\mathcal{Q}$ mapping RW converging sequences into strongly converging ones. By Proposition 6(a) we have to prove only that $\tilde{\mathcal{J}} \subset \mathcal{J}$. From Proposition 6(b) it easily follows that $\tilde{\mathcal{J}}$ is a two-sided ideal of $\mathcal{Q}$. If $K_n \in \tilde{\mathcal{J}}$ and $\|K - K_n\| \to 0$, then for every sequence $x_m \underset{\mathrm{RW}}{\to} 0$, $\|Kx_m\| < \|K_n x_m\| + \|K - K_n\| \, \|x_m\|$. Since $\|x_m\|$ is bounded, $Kx_m \underset{s}{\to} 0$; hence $K \in \tilde{\mathcal{J}}$ and $\tilde{\mathcal{J}}$ is norm closed. Let $Q \in \mathcal{Q}$ be any infinite projection and take an orthonormal sequence $x_n \underset{\mathrm{RW}}{\to} 0$ in the range of Q. Then $Qx_n = x_n \underset{s}{\nrightarrow} 0$. Thus $Q \notin \tilde{\mathcal{J}}$, which implies that $\tilde{\mathcal{J}}$ contains finite projection only; hence $\tilde{\mathcal{J}} \subset \mathcal{J}$ and thus $\tilde{\mathcal{J}} = \mathcal{J}$. Q.E.D.

Let us recall that Calkin's construction of his representation of $L(H)/\mathcal{J}$ (see **[2]**) relies essentially on Hilbert's characterization of the compact operators of $L(H)$. Theorem 7 shows that this characterization remains valid in semifinite von Neumann algebras. Thus by extending Calkin's construction to $\mathcal{Q}$ we can obtain a representation of $\mathcal{Q}/\mathcal{J}$. Most of this extension is routine adaptation of the original proofs to von Neumann algebras; therefore we omit it and refer the reader to **[2]**.

2. The generalized Wolf Property. Wolf characterized in **[8]** the operators of $L(H)$ that are not left-Fredholm as those operators A for which there is a sequence $x_n \underset{w}{\to} 0$, but $x_n \underset{s}{\nrightarrow} 0$ such that $Ax_n \underset{s}{\to} 0$.

Left-Fredholm operators are defined in general von Neumann algebras as the operators left invertible mod $\mathcal{J}$ and are shown to satisfy most of the classical properties (see **[1, 4]**). However if we replace in Wolf's Theorem the weak with the RW convergence, we obtain a characterization of a new class of $\mathcal{Q}$.

THEOREM 8. *Let $A \in \mathcal{A}$. Then the following conditions are equivalent*:

(a) *There is an infinite $P \in \mathcal{P}(\mathcal{A})$ such that $AP \in \mathcal{J}$.*

(b) *There is an orthonormal sequence $x_n \underset{RW}{\to} 0$ such that $Ax_n \underset{s}{\to} 0$.*

(c) *There is a sequence $x_n \underset{RW}{\to} 0$, but $x_n \underset{s}{\not\to} 0$, such that $Ax_n \underset{s}{\to} 0$.*

PROOF. (a) implies (b) since by Proposition 3 applied to P we can find an orthonormal sequence $x_n \underset{\mathrm{RW}}{\to} 0$ in PH, and by Proposition 6(a), $APx_n = Ax_n \underset{s}{\to} 0$.

(b) implies obviously (c). Assume (c) and assume first that $A \geqslant 0$. Let E be the spectral measure of A, let $Q_n = E[0, \frac{1}{n})$, $P_n = Q_n - Q_{n+1}$, and $P_0 = N_A = E[0]$. Then $Q_n = P_0 + \sum_{j=n}^{\infty} P_j$. If P_0 is infinite then $AP_0 = 0 \in \mathcal{J}$. Assume P_0 is finite. Then $P_0x_n \underset{s}{\to} 0$ and as $x_n \underset{s}{\not\to} 0$, we have $(I - P_0)x_n \underset{s}{\not\to} 0$. Therefore there is an $\alpha > 0$ and a subsequence x_{n_k} such that $\|(I - P_0)x_{n_k}\| > \alpha$. Let $z_k = (\|(I - P_0)x_{n_k}\|)^{-1} \cdot (I - P_0)x_{n_k}$. For every finite $P \in \mathcal{P}(\mathcal{A})$ we have $\|Pz_k\| \leqslant \frac{1}{\alpha}\|P(I - P_0)x_{n_k}\| \to 0$ as $(I - P_0)x_{n_k} \underset{\mathrm{RW}}{\to} 0$ by Proposition 6(b). Moreover

$$\|Az_k\| \leqslant \tfrac{1}{\alpha}\|A(I - P_0)x_{n_k}\| = \tfrac{1}{\alpha}\|Ax_{n_k}\| \to 0.$$

Since

$$\|Az_k\| \geqslant \|A(I - Q_m)z_k\| \geqslant \tfrac{1}{m}\|(I - Q_m)z_k\|$$

and since we can find for every m an index k_m such that $\|Az_{k_m}\| < 1/2m^2$, we have $\|(I - Q_m)z_{k_m}\| < 1/2m$. Choose k_m monotone. Moreover $\|Q_mz_{k_m}\|^2 = 1 - \|(I - Q_m)z_{k_m}\|^2 \geqslant 1 - 1/4m^2$; hence $\|Q_mz_{k_m}\| > \frac{1}{2}$. Let $y_m = (\|Q_mz_{k_m}\|)^{-1}Q_mz_{k_m}$. Then for every finite $P \in \mathcal{P}(\mathcal{A})$

$$\|Py_m\| \leqslant 2\|PQ_mz_{k_m}\| \leqslant 2\|Pz_{k_m}\| + 2\|P(I - Q_m)z_{k_m}\| \leqslant 2\|Pz_{k_m}\| + \tfrac{1}{m}.$$

As $z_{k_m} \underset{\mathrm{RW}}{\to} 0$, we have $Py_m \underset{s}{\to} 0$. Thus $y_m \in (Q_m - P_0)H$, $\|y_m\| = 1$ and $y_m \underset{\mathrm{RW}}{\to} 0$. Consider $1 = \|Q_1 y_1\|^2 = \sum_{n=1}^{\infty}\|P_ny_1\|^2$. Then there is an $N_1 < \infty$ such that $\sum_{n=1}^{N_1-1}\|P_ny_1\|^2 > \frac{1}{2}$. Since $\mathcal{A}$ is semifinite, we can find finite projections $F_n \leqslant P_n$ such that $\|F_ny_1\|^2 \geqslant \frac{1}{2}\|P_ny_1\|^2$ for $1 \leqslant n \leqslant N_1 - 1$. Thus $R_1 = \sum_{n=1}^{N_1-1} F_n \leqslant Q_1$, R_1 is finite and $\|R_1 y_1\| > \frac{1}{2}$. Consider now $1 = \|Q_{N_1} y_{N_1}\|^2 = \sum_{n=N_1}^{\infty}\|P_ny_{N_1}\|^2$. Repeat the same construction and find $N_2 < \infty$ and finite $F_n \leqslant P_n$ for $N_1 \leqslant n \leqslant N_2 - 1$ such that, for $R_2 = \sum_{n=N_1}^{N_2-1} F_n \leqslant Q_{N_1}$, $\|R_2 y_{N_1}\| > \frac{1}{2}$. Note that R_2 is finite and orthogonal to R_1. By iterating, construct a sequence of mutually orthogonal finite projections $R_k \leqslant Q_k$ such that $\|R_{k+1} y_{N_k}\| > \frac{1}{2}$. Let $R = \sum_{k=1}^{\infty} R_k$. Since $\|Ry_{N_k}\| \geqslant \|R_{k+1} y_{N_k}\| > \frac{1}{2}$, we have $Ry_n \underset{s}{\not\to} 0$. Hence $R \notin \mathcal{J}$ and thus R is infinite. On the other hand $R = \sum_{n=1}^{\infty} F_n$ where $F_n \leqslant P_n$ are finite and as

$$\left\|AR - A\sum_{n=1}^{j-1} F_n\right\| = \left\|A\sum_{n=j}^{\infty} F_n\right\| \leqslant \|AQ_j\| \leqslant \frac{1}{j}$$

we see that AR can be approximated in norm by the compact operators $A\sum_{n=1}^{j-1} F_n$; hence $AR \in \mathcal{J}$. Finally note that if A is not positive then we can consider $|A| = (A^*A)^{1/2}$. Then $A = U|A|$ and since $\|Ax_n\| = \|\,|A|x_n\| \to 0$ we can apply the result obtained above to $|A|$ and find an infinite $R \in \mathcal{P}(\mathcal{A})$ such that $|A|R \in \mathcal{J}$. Then $AR = U|A|R \in \mathcal{J}$. Q.E.D.

Let us denote by $\mathfrak{F}_+$ the left-Fredholm class and by $\tilde{\mathfrak{F}}_+$ the complement of the class characterized by Theorem 8, i.e., the class $\tilde{\mathfrak{F}}_+ = \{A \in \mathcal{Q}: P \in \mathcal{P}(\mathcal{Q}), AP \in \mathcal{J} \Rightarrow P \in \mathcal{J}\}$.

In $L(H)$, $\mathfrak{F}_+ = \tilde{\mathfrak{F}}_+$ (Wolf Theorem [8]); however for general von Neumann algebras $\mathfrak{F}_+ \subset \tilde{\mathfrak{F}}_+$ and the inclusion is proper unless $\mathcal{Q}$ is semifinite and has a nonlarge center (see [5]). Since $\tilde{\mathfrak{F}}_+$ satisfies all the "algebraic" properties of $\mathfrak{F}_+$, we call it the almost left-Fredholm class and we further study it in [5].

Remark 9. In this paper we were interested in treating compact elements of a von Neumann algebra in a Hilbert space operator theoretic way. However, both the compact and Fredholm (or almost Fredholm) classes are defined without reference to a Hilbert space representation and thus we can reformulate Theorems 7 and 8 in a space-free way. Let $\mathcal{Q}_*$ be the predual of a semifinite von Neumann algebra $\mathcal{Q}$. Let us call RW the topology $\sigma((\mathcal{Q}_*)_1, \mathcal{P}(\mathcal{Q}) \cap \mathcal{J})$ on the unit ball of $\mathcal{Q}_*$. Then we have

Theorem 7′. *$A \in \mathcal{J}$ iff $f_n(A) \to f(A)$ for every sequence $f_n \in (\mathcal{Q}_*)_1$ such that $f_n \underset{RW}{\to} f$.*

Theorem 8′. *$A \notin \tilde{\mathfrak{F}}_+$ iff there is a sequence $f_n \in (\mathcal{Q}_*)_1^+$ such that $\|f_n\| = 1$, $f_n \underset{RW}{\to} 0$ and $f_n(A^*A) \to 0$.*

These theorems can be proven with the same techniques as Theorems 7 and 8: use Proposition 3 by noting that if $\mathcal{Q}$ has a representation on the Hilbert space H, then $x_n \in (H)_1$ and $x_n \underset{\mathrm{RW}}{\to} 0$ iff $\omega_{x_n} \in (\mathcal{Q}_*)_1^+$ and $\omega_{x_n} \underset{\mathrm{RW}}{\to} 0$.

Bibliography

1. M. Breuer, *Fredholm theories in von Neumann algebras*. I, Math. Ann. **178** (1968), 243–254.

2. J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. **42** (1941), 839–873.

3. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, 2nd ed., Gauthier-Villars, Paris, 1969.

4. V. Kaftal, *On the theory of compact operators in von Neumann algebras*. I, Indiana Univ. Math. J. **26** (1977), 447–457.

5. ______, *Almost Fredholm operators in von Neumann algebras*, Integral Equations Operator Theory (to appear).

6. C. Peligrad and L. Zsido′, *A Riesz decomposition theorem in W^*-algebras*, Acta Sci. Math. (Szeged) **34** (1973), 317–322.

7. M. Sonis, *On a class of operators in von Neumann algebras with Segal measures*, Math. USSR-Sb. **13** (1971), 344–359.

8. F. Wolf, *On the invariance of the essential spectrum under a change of boundary conditions of partial differential operators*, Nederl. Akad. Wetensch. Indag. Math. **21** (1959), 142–315.

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221