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ON A THEOREM OF BAKER, LAWRENCE AND ZORZITTO

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ABSTRACT. The result of J. Baker, J. Lawrence and F. Zorzitto on the stability of the equation $f(x+y)=f(x)f(y)$ is generalized by proving the following theorem: if G is a semigroup and V is a right invariant linear space of complex valued functions on G, and if f, m are complex valued functions on G for which the function $x \to f(xy)-f(x)m(y)$ belongs to V for every y in G, then either f is in V or m is exponential.

In [1] J. Baker, J. Lawrence and F. Zorzitto, solving a problem of E. Lukacs on the stability of the functional equation $f(x+y)=f(x)f(y)$ proved that if f is a function from a vector space to the real numbers satisfying

$$|f(x+y)-f(x)f(y)|<\delta$$

then f is either bounded or exponential. This result was also generalized and simplified in [2]. Here we generalize this result in another way.

Let G be a semigroup and V be a linear space of complex valued functions on G. Then V is called right invariant if f belongs to V implies that the function $x \to f(xy)$ belongs to V for every y in G. Similarly, we can define left invariant linear spaces, and we call V invariant if it is right and left invariant.

The complex valued function $m: G \to \mathbf{C}$ ($\mathbf{C}$ denotes the set of complex numbers) is called an exponential if for every x, y in G we have

$$m(xy)=m(x)m(y).$$

Our main result is the following

THEOREM. *Let G be a semigroup and V be a right invariant linear space of complex valued functions on G. Let f, m: $G \to \mathbf{C}$ be complex valued functions for which the function $x \to f(xy)-f(x)m(y)$ belongs to V for every y in G. Then either f is in V or m is an exponential.*

PROOF. Suppose that m is not an exponential. Then there exist y, z in G with the property $m(yz)-m(y)m(z)\neq 0$. On the other hand we have, for all x in G,

$$f(xyz)-f(xy)m(z)=[f(xyz)-f(x)m(yz)]-m(z)[f(xy)-f(x)m(y)] \\ +f(x)[m(yz)-m(y)m(z)]$$

and hence

$$f(x)=[(f(xyz)-f(xy)m(z))-(f(xyz)-f(x)m(yz)) \\ +m(z)(f(xy)-m(y)f(x))]\cdot[m(yz)-m(y)m(z)]^{-1}.$$

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Here the right-hand side as a function of x belongs to V, and hence so does f.

COROLLARY. *Let G be a semigroup with identity and V be an invariant linear space of complex valued functions on G. Let f, m: $G \to \mathbf{C}$ be complex valued functions for which the functions $x \to f(xy) - f(x)m(y)$ and $y \to f(xy) - f(x)m(y)$ belong to V for every y in G and x in G, respectively. Then either f is in V or m is an exponential and $f = f(1)m$.*

PROOF. Suppose that f is not in V. Then by the preceding theorem, m is an exponential. On the other hand, the function $y \to f(y) - f(1)m(y)$ is in V and for x, y in G we have

$$\begin{aligned} f(xy) - f(1)m(xy) &= f(xy) - f(x)m(y) + f(x)m(y) - f(1)m(x)m(y) \\ &= f(xy) - f(x)m(y) + [f(x) - f(1)m(x)]m(y). \end{aligned}$$

If there is an x_0 in G for which $f(x_0) \neq f(1)m(x_0)$, then m belongs to V and so does f, which is a contradiction. Hence $f = f(1)m$ which was to be proved.

REMARK. Here we make clear how the corollary generalizes the cited result. Let G be an Abelian semigroup with identity and V be the space of bounded complex valued functions on G. If f, m: $G \to \mathbf{C}$ are functions for which there exist M_1, M_2: $G \to [0, +\infty)$ such that

$$|f(xy) - f(x)m(y)| \leq \min(M_1(x), M_2(y))$$

for all x, y in G then either f is bounded or m is exponential and $f = f(1)m$.

EXAMPLE. Let G be a commutative topological group and f: $G \to \mathbf{C}$ be such that $x \to f(x + y) - f(x)f(y)$ is continuous for each y in G. Then either f is continuous or exponential. Other interesting examples can be constructed by taking V to be the class of measurable or integrable functions on appropriate groups.

REFERENCES

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