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PROOF OF A CONJECTURE OF ERDÖS ABOUT THE LONGEST POLYNOMIAL

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ABSTRACT. In 1939 P. Erdös conjectured that the Chebyshev polynomial $T_n(x)$ has a maximal arc-length in $[-1, 1]$ among the polynomials of degree n which are bounded by 1 in $[-1, 1]$. We prove this conjecture for every natural n.

1. Introduction. P. Erdös proved in [2] that the function $\cos nt$ has a maximal arc-length in $[-\pi, \pi]$ among all trigonometric polynomials of order n with a uniform norm equal to 1. He has conjectured that the Chebyshev polynomial

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leqslant x \leqslant 1,$$

is the unique extremal function in the corresponding analogous problem in the set π_n of algebraic polynomials of degree less than or equal to n.

Denote by $l(f)$ the arc-length of the function f in $[-1, 1]$, i.e.,

$$l(f) := \int_{-1}^{1}\left[1 + f'^2(x)^2\right]^{1/2} dx.$$

Set $\|f\| = \max\{|f(x)|: -1 \leqslant x \leqslant 1\}$.

CONJECTURE OF ERDÖS. The quantity

$$\sup\{l(f): f \in \pi_n, \|f\| \leqslant 1\} \qquad (n = 1, 2, \ldots)$$

is attained if and only if $f = \pm T_n$.

This conjecture has remained an open problem for over 40 years. In a recent work Szabados [4] showed that the polynomials T_n are asymptotically extremal as $n \to \infty$. We prove here the conjecture of Erdös for each natural number n. Our proof is based on a variational approach.

2. Explanatory statement. The problem of Erdös is set for the domain $[-1, 1] \times [-1, 1]$, i.e., for the class of polynomials $f \in \pi_n$ such that $|f(x)| \leqslant 1$ if $|x| \leqslant 1$. One may guess that the solution $f(x)$ in this particular case suffices to construct the solution $f(M; x)$ of the corresponding problem about the longest polynomial in the domain $[-1, 1] \times [-M, M]$ for every $M > 0$. One even suggests the following simple formula:

$$(*) \qquad f(M; x) = Mf(x).$$

It turns out (see Theorem 1) that $(*)$ is actually true. But this is not evident. The problem $(*)$ is as difficult as that of Erdös. In any case, the relatioin $(*)$ yields easily the conjecture of Erdös. Indeed, suppose that $(*)$ holds for every $M > 0$.

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Then

$$\frac{1}{M}\int_{-1}^{1}\left[1 + M^2 g'^2(x)\right]^{1/2} dx \leqslant \frac{1}{M}\int_{-1}^{1}\left[1 + M^2 f'^2(x)\right]^{1/2} dx$$

for each $M > 0$, provided $g \in \pi_n$, and $\|g\| \leqslant 1$. If we let M tend to infinity, we get $\int_{-1}^{1} |g'(x)|\, dx \leqslant \int_{-1}^{1} |f'(x)|\, dx$. Thus, f should have a maximal variation in $[-1, 1]$. Therefore $f = \pm T_n$.

Finally, note that the problem on an arbitrary interval $[a, b]$ is easily reduced to the problem on $[-1, 1]$ by a linear transformation.

3. Main result. In what is to follow, let M be a fixed positive number. With every natural number n we associate the set $\Omega_n \subset \pi_n$ which is defined as follows. The polynomial $f \in \pi_n$ belongs to Ω_n if there exist $m + 1$ points $\{x_i\}_0^m$ $(m \in \{1, \ldots, n\})$ such that

$$-1 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1,$$
$$|f(x_i)| = M, \qquad i = 0, \ldots, m,$$
$$f(x_i) = -f(x_{i+1}), \qquad i = 0, \ldots, m-1$$

and $f(x)$ is a monotone function in $[x_i, x_{i+1}]$, $i = 0, \ldots, m - 1$. It is clear that $\|f\| = M$ if $f \in \Omega_n$.

The basic idea of our proof is presented in the following lemma.

LEMMA 1. *Suppose that* $f \in \pi_n$, $\|f\| = M$ *and*

$$l(f) = \sup\{l(g) \colon g \in \pi_n,\ \|g\| \leqslant M\}.$$

Then $f \in \Omega_n$.

PROOF. Without loss of generality we assume that $f(x) > 0$ for each sufficiently large $x > 0$. Denote by $\{x_i\}_1^{m-1}$ the distinct zeros of $f'(x)$ in $(-1, 1)$. Obviously $m \leqslant n$. Set, for convenience, $x_0 = -1$, $x_m = 1$, $\omega(x) = f'(x)$. We shall show that

$$f(x_i) = (-1)^{m-1} M, \qquad i = 0, \ldots, m. \tag{1}$$

This implies that $f \in \Omega_n$.

Introduce the polynomials

$$g_i(x) = (x^2 - 1)\omega(x)/(x - x_i), \qquad i = 0, \ldots, m.$$

We intend to estimate the arc-length $\sigma_i(\varepsilon) := l(f + \varepsilon g_i)$ for small ε. Our first task is to show that

$$\sigma_i'(0) > 0 \tag{2}$$

for $i = 0, \ldots, m$. It is seen that

$$\sigma_i'(0) = \int_{-1}^{1} \frac{\omega(x)}{\left[1 + \omega^2(x)\right]^{1/2}} g_i'(x)\, dx.$$

In the case $i = 0$ a straightforward calculation gives

$$\sigma_0'(0) = 2\left[1 + \omega^2(-1)\right]^{1/2} - \int_{-1}^{1}\left[1 + \omega^2(x)\right]^{-1/2} dx > 0.$$

Similarly, $\sigma_m'(0) > 0$. Now suppose that $1 \leqslant i \leqslant m-1$. Integrating by parts, we get

$$\sigma_i'(0) = \int_{-1}^{1} \frac{x^2-1}{x-x_i}\left\{\left[1+\omega^2(x)\right]^{-1/2}\right\}' dx.$$

The integrand is a continuous function in $[-1, 1]$. Therefore $\sigma_i'(0) < \infty$ and $\sigma_i'(0) = \lim\{\mathfrak{T}_i(\delta): \delta \to 0\}$ where

$$\mathfrak{T}_i(\delta) = \int_{\Omega(\delta)} \frac{x^2-1}{x-x_i}\left\{\left[1+\omega^2(x)\right]^{-1/2}\right\}' dx$$

and $\Omega(\delta) := [-1, x_i - \delta] \cup [x_i + \delta, 1]$. Next we calculate $\mathfrak{T}_i(\delta)$. Observe first that $\omega(x_i \pm \delta) = O(\delta)$. This yields, for instance, by Taylor's formula, that

$$\left[1+\omega^2(x_i \pm \delta)\right]^{-1/2} = 1 + O(\delta^2). \tag{3}$$

Further, by the mean-value theorem for integrals, there exist points $\xi_1 = \xi_1(\delta) \in [-1, x_i - \delta]$ and $\xi_2 = \xi_2(\delta) \in [x_i + \delta, 1]$ such that

$$\begin{aligned} &\int_{-1}^{x_i-\delta}\left[1+\omega^2(x)\right]^{-1/2}(x-x_i)^{-2}\,dx = c_1(\delta)\left[1/\delta - 1/(1+x_i)\right],\\ &\int_{x_i+\delta}^{1}\left[1+\omega^2(x)\right]^{-1/2}(x-x_i)^{-2}\,dx = c_2(\delta)\left[1/\delta - 1/(1-x_i)\right] \end{aligned} \tag{4}$$

where $c_j(\delta) = [1 + \omega^2(\xi_j)]^{-1/2}$, $j = 1, 2$. Obviously

$$0 < c_j(\delta) \leqslant 1, \qquad j = 1, 2. \tag{5}$$

Let us set, for convenience, $A(\delta) = \int_{\Omega(\delta)}[1 + \omega^2(x)]^{-1/2}\,dx$. Now, taking into account the relations (3) and (4), after integration by parts, we obtain

$$\begin{aligned} \mathfrak{T}_i(\delta) &= \left[(x^2-1)/(x-x_i)\right]\left[1+\omega^2(x)\right]^{-1/2}\Big|_{x_i+\delta}^{x_i-\delta}\\ &\quad -\int_{\Omega(\delta)}\left[1+\omega^2(x)\right]^{-1/2}\left\{1+\left(1-x_i^2\right)/(x-x_i)^2\right\}dx\\ &= \delta^{-1}\left[c_1(\delta)+c_2(\delta)-2\right]\left(x_i^2-1\right) + O(\delta) - A(\delta)\\ &\quad - c_1(\delta)(x_i-1) + c_2(\delta)(x_i+1). \end{aligned}$$

But, as we have already mentioned, $\mathfrak{T}_i(\delta)$ has a limit as $\delta \to 0$. Then $c_1(\delta) + c_2(\delta)$ must tend to 2, which combined with (5) implies $c_j(\delta) \to 1$, $j = 1, 2$, as $\delta \to 0$. Moreover, $c_j(\delta) = 1 - \alpha_j\delta + o(\delta)$, $j = 1, 2$, with some constants $\alpha_j \geqslant 0$. Therefore

$$\sigma_i'(0) = \lim\{\mathfrak{T}_i(\delta): \delta \to 0\} = -A(0) + 2 - (\alpha_1 + \alpha_2)\left(x_i^2 - 1\right) > 0.$$

Our claim (2) is proved.

Now, let us assume that f does not belong to Ω_n. Then there exists at least one point $x_i \in \{x_0, \ldots, x_m\}$ such that $|f(x_i)| < M$. Consider the polynomial $\varphi_\varepsilon(x) := f(x) + \varepsilon g_i(x)$. Evidently, $l(\varphi_\varepsilon) = \sigma_i(\varepsilon) = \sigma_i(0) + \varepsilon\sigma_i'(t_\varepsilon) = l(f) + \varepsilon\sigma_i'(t_\varepsilon)$ where $0 < t_\varepsilon < \varepsilon$. But, according to (2), there eixsts an $\varepsilon_0 > 0$ such that $\sigma := \min\{\sigma_i'(t): 0 \leqslant t \leqslant \varepsilon_0\} > 0$. Therefore

$$l(\varphi_\varepsilon) \geqslant l(f) + \sigma\varepsilon \tag{6}$$

for each $\varepsilon \in [0, \varepsilon_0]$.

Let us estimate the uniform norm of φ_ε in $[-1, 1]$ for small ε. In order to do this, it suffices to investigate the function $\varphi_\varepsilon(x)$ near the points $\{x_j\}$ for which $|f(x_j)| = M$. Let x_k be such a point. Without loss of generality we may assume that $f(x_k) = M$. Suppose that h is chosen to satisfy the requirement $x_j \notin [x_k - h, x_k + h] \cap [-1, 1] =: B(x_k; h)$ for every $j \neq k$. Let $\varphi_\varepsilon(x)$ attain its maximal value in the neighbourhood $B(x_k; h)$ of x_k at the point $z_k(\varepsilon)$. On expanding $\varphi_\varepsilon(x)$ in a partial Taylor series around $x = x_k$, we get

$$\varphi_\varepsilon(z_k(\varepsilon)) \leqslant M + \varepsilon \| g_i' \| \, |z_k(\varepsilon) - x_k|$$

for sufficiently small $\varepsilon > 0$. It is not difficult to see that $|z_k(\varepsilon) - x_k| \to 0$ as $\varepsilon \to 0$. Then, in view of the last inequality, $\|\varphi_\varepsilon\| \leqslant M + \varepsilon\delta(\varepsilon)$, where $\delta(\varepsilon)$ is a function which tends to zero as $\varepsilon \to 0$. Now construct the polynomial

$$\psi_\varepsilon(x) = \left(1 - \frac{\varepsilon\delta(\varepsilon)}{M + \varepsilon\delta(\varepsilon)}\right)\varphi_\varepsilon(x).$$

Clearly, $\psi_\varepsilon \in \pi_n$ and $\|\psi_\varepsilon\| \leqslant M$. We shall show that $l(\psi_\varepsilon) > l(f)$ for small $\varepsilon > 0$. Indeed, since $L := \partial l(\lambda f)/\partial\lambda|_{\lambda=1} > 0$, we have $l(\psi_\varepsilon) > l(\varphi_\varepsilon) - (2L/M)\varepsilon\delta(\varepsilon)$ for small $\varepsilon > 0$. Next we apply (6) and get

$$l(\psi_\varepsilon) > l(f) + [\sigma - (2L/M)\delta(\varepsilon)]\varepsilon > l(f)$$

for sufficiently small $\varepsilon > 0$. Thus, f is not extremal, a contradiction. Therefore $|f(x_i)| = M$ for $i = 0, \dots, m$. Since $\{x_i\}_1^{m-1}$ are all distinct zeros of $f'(x)$ in $(-1, 1)$, we conclude that (1) is valid. The lemma is proved.

It remains to show that the extremal polynomial f must have $n + 1$ points of alternation. For this, we give below an interesting property of the Chebyshev polynomial $T_n(x)$.

Let $\{\theta_k\}_0^n$ be the extremal points of $T_n(x)$ in $[-1, 1]$. It is well known (see Rivlin [3]) that $\theta_0 = -1$, $\theta_n = 1$ and $T_n(\theta_k) = (-1)^{n-k}$, $k = 0, \dots, n$. Suppose that $f \in \Omega_n$ and $f'(x)$ has $m - 1$ distinct zeros $x_1, \dots, x_{m-1}$ in $(-1, 1)$. Evidently, there is an $i \in \{0, \dots, m-1\}$ such that $x_i < 0 \leqslant x_{i+1}$. Consider the partition of $[-1, 1]$ into subintervals $[x_0, x_1], \dots, [x_i, 0]$, $[0, x_{i+1}], \dots, [x_{m-1}, x_m]$ which we denote, for simplicity, by $I_0, \dots, I_m$, respectively. Define the points t_1 and t_2 by the conditions

$$t_1 \in [\theta_i, \theta_{i+1}], \qquad MT_n(t_1) = f(0),$$
$$t_2 \in [\theta_{i+n-m}, \theta_{i+n-m+1}], \qquad MT_n(t_2) = f(0).$$

Denote the intervals $[\theta_0, \theta_1], \dots, [\theta_i, t_1]$, $[t_2, \theta_{i+n-m+1}], \dots, [\theta_{n-1}, \theta_n]$ by $I_0^*, \dots, I_m^*$. We shall refer to I_k^* as the corresponding interval to I_k.

LEMMA 2. *Suppose that f is a polynomial from the set Ω_n with $m + 1$ extremal points, $\alpha \in (-M, M)$ and $k \in \{0, \dots, m\}$. Let the points ξ and η satisfy the conditions*

$$\xi \in I_k^*, \quad MT_n(\xi) = \alpha, \qquad \eta \in I_k, \quad f(\eta) = \alpha.$$

Then $|f'(\eta)| \leqslant M|T_n'(\xi)|$.

The assertion follows easily from a known extremal property of cos nt. The proof is given with details in [1].

We are now prepared to prove the main theorem.

THEOREM 1. *Let n be an arbitrary natural number. Then, for each $M > 0$, the quantity*

$$\sup\{l(f): f \in \pi_n, \|f\| \leqslant M\}$$

is attained if and only if $f = \pm MT_n$.

PROOF. Note first that the inequality $|d| \leqslant |c|$ implies

$$(1 + c^2)^{1/2} \leqslant (1 + d^2)^{1/2} + |c| - |d|. \tag{7}$$

We shall make use of this in the sequel. Suppose that $f \in \Omega_n$ and $[-1, 1] = I_0 \cup \cdots \cup I_m$ is the partition of $[-1, 1]$ induced by f. Let the intervals $I = [z_1, z_2]$ and $I^* = [z_1^*, z_2^*]$ be corresponding. Denote by $u(y)$ and $v(y)$ the inverse functions of $f(x)$ and $MT_n(x)$ in I and I^*, respectively. According to Lemma 2, we have $|v'(y)| \leqslant |u'(y)|$ for each $y \in (-M, M)$. Then, applying (7), we get

$$\int_{-M}^{M}\left[1 + u'^2(y)\right]^{1/2} dy \leqslant \int_{-M}^{M}\left[1 + v'^2(y)\right]^{1/2} dy + \int_{-M}^{M}|u'(y)|\, dy - \int_{-M}^{M}|v'(y)|\, dy.$$

Denote by $l(g; K)$ the arc-length of g over the set K. Then the above inequality means that $l(f; I) \leqslant l(MT_n; I^*) + |z_2 - z_1| - |z_2^* - z_1^*|$. Summing for $I = I_0, \ldots, I_m$, we obtain

$$l(f) \leqslant l\left(MT_n; [-1, t_1] \cup [t_2, 1]\right) + t_2 - t_1 \leqslant l(MT_n).$$

The equality holds if and only if $t_1 = t_2$, i.e., iff $f = \pm MT_n$. The theorem is proved.

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