

ON COISOTROPIC IMBEDDINGS OF PRESYMPLECTIC MANIFOLDS

MARK J. GOTAY

ABSTRACT. Existence and uniqueness theorems are proved for coisotropic imbeddings of presymplectic manifolds into symplectic manifolds.

I. Introduction. A manifold M is *presymplectic* if it carries a distinguished closed 2-form ω of constant rank; if ω is nondegenerate, then (M, ω) is *symplectic*. Let (M, ω) be a presymplectic manifold. A *coisotropic imbedding* of (M, ω) into a symplectic manifold (P, Ω) is a closed imbedding $j: M \rightarrow P$ such that

- (i) $j^*\Omega = \omega$,
- (ii) $TM^\perp \subseteq Tj(TM)$.

Such imbeddings play an important role in both the theory of constraints and quantization theory [1], [2], where (M, ω) represents the (degenerate) phase space of a physical system.

We show that every presymplectic manifold may be coisotropically imbedded in a symplectic manifold. Specifically, we prove the following

EXISTENCE THEOREM. *There exists a symplectic structure on a neighborhood of the zero-section of E^* , where E denotes the characteristic bundle of (M, ω) . Moreover, (M, ω) may be coisotropically imbedded in this neighborhood as the zero-section.*

Once existence has been established, we classify symplectic neighborhoods of coisotropic imbeddings. To this end it is useful to introduce, following Weinstein [3]–[5], the notion of a *neighborhood equivalence* of two imbeddings

$$j_1: M \rightarrow (P_1, \Omega_1) \quad \text{and} \quad j_2: M \rightarrow (P_2, \Omega_2).$$

This consists of

- (i) open neighborhoods U_i of $j_i(M)$ in P_i ,
- (ii) a symplectomorphism $\psi: (U_1, \Omega_1) \rightarrow (U_2, \Omega_2)$ such that $\psi \circ j_1 = j_2$.

We then prove a

LOCAL UNIQUENESS THEOREM. *All coisotropic imbeddings of (M, ω) are neighborhood equivalent.*

Thus, up to local symplectomorphism about M , there is a unique extension of (M, ω) to a symplectic manifold in which M is coisotropic.

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The Local Uniqueness Theorem combined with the Existence Theorem shows that *there exists a symplectic structure on a neighborhood of the zero-section of E^* such that every symplectic manifold containing (M, ω) as a coisotropic submanifold is, near M , symplectomorphic to this neighborhood.* The theorems in this article thus provide a complete local characterization of coisotropic imbeddings of presymplectic manifolds into symplectic manifolds.

These results are complementary to those obtained by Weinstein [3], [4] regarding isotropic imbeddings. In particular, the isotropic and coisotropic cases coincide when $\omega = 0$, in which case M is to be imbedded as a Lagrangian submanifold. Since in this instance $E^* = T^*M$, the two theorems imply Weinstein's result [3], [5] that every symplectic manifold containing M as a Lagrangian submanifold is, near M , symplectomorphic to a neighborhood of the zero-section in T^*M .

Both of the theorems rely heavily upon Weinstein's extended version of the Darboux theorem [3], [5] which for convenience I state here.

EXTENSION THEOREM. *Let P be a manifold, $M \subseteq P$ a closed submanifold.*

(A) *Suppose that $(T_M P, \Omega)$ is a symplectic vector bundle and that $\Omega|_{TM}$ is closed. Then Ω extends to a symplectic structure on a neighborhood of M in P .*

(B) *Let Ω_1 and Ω_2 be symplectic forms on P_1 and P_2 respectively and let M be imbedded in both P_1 and P_2 . Suppose that $(T_M P_1, \Omega_1)$ and $(T_M P_2, \Omega_2)$ are isomorphic as symplectic vector bundles. Then there are neighborhoods U_i of M in P_i and a symplectomorphism $\psi: (U_1, \Omega_1) \rightarrow (U_2, \Omega_2)$ such that $\psi|_M = \text{id}_M$.*

Notation and terminology are explained in [3].

II. Proof of the Existence Theorem. Let (M, ω) be presymplectic, and consider the vector bundle $\pi: E \rightarrow M$ with fiber

$$E_m = \{x \in T_m M | \omega(x) = 0\}.$$

E is the characteristic bundle of (M, ω) ; denote by E^* the dual bundle.

Imbed M as the zero-section of E^* . The strategy is to make $T_M E^*$ into a symplectic vector bundle over M and then apply (A) of the Extension Theorem.

The restricted tangent bundle $T_M E^*$ has the canonical decomposition

$$(1) \quad T_M E^* = TM \oplus E^*$$

and, since E is a subbundle of TM , one may further split

$$(2) \quad TM = G \oplus E.$$

Denote by pr the induced projection $T_M E^* \rightarrow E \oplus E^*$ or $m \in M$, let $\omega_E(m)$ denote the canonical symplectic structure on $E_m \oplus E_m^*$,

$$(3) \quad \omega_E(m)(e \oplus e^*, f \oplus f^*) = f^*(e) - e^*(f).$$

Set

$$(4) \quad \Omega_G = \pi^* \omega + \omega_E \circ (pr \times pr).$$

Ω_G is clearly smooth, well defined and nondegenerate.

Thus $(T_M E^*, \Omega_G)$ is a symplectic vector bundle. Since $\Omega_G|_{TM} = \omega$ is closed, (A) of the Extension Theorem applies. Consequently, Ω_G extends to a symplectic form on a neighborhood of M in E^* which clearly pulls back to ω on M .

It remains to show that M is a coisotropic submanifold of E^* , i.e., that $TM^\perp \subseteq TM$ in $T_M E^*$. But this follows immediately from (4). Thus the Existence Theorem is proved.

III. Proof of the Local Uniqueness Theorem. Let (M, ω) be coisotropically imbedded in (P, Ω) , so that $E = TM^\perp \subseteq TM$. Consider the symplectic vector bundle $(T_M P, \Omega)$, where Ω is the restriction to $T_M P$ of the symplectic structure on P , and let G be any complement (2) of E in TM . Since G is a symplectic subbundle of $(T_M P, \Omega)$ so is G^\perp . Hence one has the splitting

$$(5) \quad T_M P = G \oplus G^\perp.$$

Now $E \subseteq G^\perp$ and, as $E^\perp \cap G^\perp = (E \oplus G)^\perp = E$, it follows that E is a Lagrangian subbundle of G^\perp . Consequently [3], there exists a symplectomorphism

$$(G^\perp, \Omega|_{G^\perp}) \approx (E \oplus E^*, \omega_E),$$

where ω_E is given by (3). Combining this result with (5) one obtains, mimicking (1), a vector bundle isomorphism

$$(6) \quad T_M P \approx G \oplus E \oplus E^*.$$

Let Ω_G be the pullback of Ω to $G \oplus E \oplus E^*$ by the isomorphism (6). The symplectic structure Ω_G satisfies, and is in fact completely characterized by, the following three properties:

- (i) $\Omega_G|(G \oplus E) = \omega$.
- (ii) $\Omega_G|(E \oplus E^*) = \omega_E$.
- (iii) $G^\perp = E \oplus E^*$.

It follows that Ω_G is given explicitly by (4), where $\pi: G \oplus E \oplus E^* \rightarrow G \oplus E$ and $pr: G \oplus E \oplus E^* \rightarrow E \oplus E^*$ are the projections.

Consequently, the symplectic vector bundle $(G \oplus E \oplus E^*, \Omega_G)$ depends only upon M, ω and the decomposition (2) of TM and thus is independent of the imbedding space (P, Ω) . If (P', Ω') is another symplectic manifold in which (M, ω) is coisotropically imbedded, then, the prior constructions give rise to vector bundle symplectomorphisms

$$(T_M P, \Omega) \approx (G \oplus E \oplus E^*, \Omega_G) \approx (T_M P', \Omega')$$

for some fixed (but irrelevant) splitting (2). We therefore conclude from (B) of the Extension Theorem that the coisotropic imbeddings $(M, \omega) \rightarrow (P, \Omega)$ and $(M, \omega) \rightarrow (P', \Omega')$ are neighborhood equivalent. Thus there is but one neighborhood equivalence class of coisotropic imbeddings of (M, ω) .

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA,
CANADA T2N 1N4