

KILLING VECTOR FIELDS ON COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT. We discuss Killing vector fields with finite global norms on complete Riemannian manifolds whose Ricci curvatures are nonpositive or negative.

1. It is well known that if a compact Riemannian manifold has nonpositive Ricci curvature then every Killing vector field is a parallel vector field (cf. [3]). In this note, we discuss Killing vector fields with finite global norms on complete Riemannian manifolds. One of our results is that if M is a complete Riemannian manifold with nonpositive Ricci curvature then every Killing vector field on M with finite global norm is a parallel vector field. This is a generalization of the above well-known result. We also discuss the volume of a complete noncompact Riemannian manifold with nonpositive Ricci curvature. Our ideas are based on those of the papers of A. Andreotti and E. Vesentini [1] and, especially, H. Kitahara [2].

We shall be in the C^∞ -category. The manifolds considered are connected and orientable. The indices h, i, j, k, \dots run over the range $\{1, 2, \dots, n\}$ and the Einstein summation convention will be used.

2. Let M be an n -dimensional complete Riemannian manifold and g (resp. ∇) its Riemannian metric tensor field (resp. its Levi-Civita connection). Let $\{U: (x^1, \dots, x^n)\}$ denote a local coordinate system on M . g_{ij} denotes the components of g and (g^{ij}) denotes the inverse matrix of the matrix (g_{ij}) . We set $\nabla_i = \nabla_{\partial/\partial x^i}$ and $\nabla^i = g^{ij}\nabla_j$.

Let $\Lambda^s(M)$ be the space of all s -forms on M and $\Lambda_0^s(M)$ the subspace of $\Lambda^s(M)$ composed of forms with compact supports. $\eta \in \Lambda^s(M)$ may be expressed locally as

$$\eta = (1/s!) \eta_{i_1 \dots i_s} dx^{i_1} \wedge \dots \wedge dx^{i_s}.$$

Let \langle , \rangle denote the local scalar product; the global scalar product $\langle\langle , \rangle\rangle$ is defined by

$$\langle\langle \xi, \eta \rangle\rangle = \int_M \langle \xi, \eta \rangle * 1 = \int_M \xi \wedge * \eta$$

for any $\xi, \eta \in \Lambda_0^s(M)$, where $*$ denotes the star operator (cf. [4]). Let $L_2^s(M)$ be the completion of $\Lambda_0^s(M)$ with respect to the scalar product $\langle\langle , \rangle\rangle$. $d: \Lambda^s(M) \rightarrow \Lambda^{s+1}(M)$ denotes the exterior derivative and $\delta: \Lambda^s(M) \rightarrow \Lambda^{s-1}(M)$ is defined by

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$\delta = (-1)^{s+n+1} * d *$. Then we have $\langle\langle d\xi, \eta \rangle\rangle = \langle\langle \xi, \delta\eta \rangle\rangle$ for any $\xi \in \Lambda_0^s(M)$, $\eta \in \Lambda_0^{s+1}(M)$. The Laplacian operator Δ acting on $\Lambda^*(M) = \sum_s \Lambda^s(M)$ is defined by $\Delta = d\delta + \delta d$.

For $\xi \in \Lambda^1(M)$, we have

$$(2.1) \quad (d\xi)_{ij} = \nabla_i \xi_j - \nabla_j \xi_i,$$

$$(2.2) \quad \delta \xi = -\nabla^i \xi_i,$$

$$(2.3) \quad (\Delta \xi)_i = -\nabla^j \nabla_j \xi_i - R_i^{h*} \xi_h,$$

where R_i^{h*} denotes the components of the Ricci tensor field of ∇ (cf. [4]).

Hereafter, we identify the vector fields and its dual 1-forms with respect to g and they are represented by the same letters. For a vector field $\xi = \xi^i \partial/\partial x^i$, we have its dual 1-form $\xi = \xi_j dx^j = g_{ij} \xi^i dx^j$.

A vector field ξ on M is called a Killing vector field if $\mathcal{L}_\xi g = 0$ where \mathcal{L} denotes the Lie derivative operator. A Killing vector field ξ satisfies the following:

$$(2.4) \quad \nabla_i \xi_j + \nabla_j \xi_i = 0,$$

and, from this, we have

$$(2.5) \quad \nabla^i \xi_i = 0.$$

A Killing vector field on M is called “with finite global norm” if its dual 1-form with respect to g belongs in $L_2^1(M) \cap \Lambda^1(M)$.

3. Let o be a point of M and fix it. For each point $p \in M$, we denote by $\rho(p)$ the geodesic distance from o to p . Let $B(\alpha) = \{p \in M | \rho(p) < \alpha\}$ for $\alpha > 0$. We choose a C^∞ -function μ on \mathbf{R} (the reals) satisfying

- (i) $0 \leq \mu(t) \leq 1$ on \mathbf{R} ,
- (ii) $\mu(t) = 1$ for $t \leq 1$,
- (iii) $\mu(t) = 0$ for $t \geq 2$,

and we set

$$w_\alpha(p) = \mu(\rho(p)/\alpha)$$

for $\alpha = 1, 2, 3, \dots$. Then we have

LEMMA 1 (cf. [1]). *There exists a positive number A , depending only on μ , such that*

- (i) $\|dw_\alpha \wedge \xi\|_{B(2\alpha)}^2 \leq (nA/\alpha^2) \|\xi\|_{B(2\alpha)}^2$,
- (ii) $\|dw_\alpha \wedge *\xi\|_{B(2\alpha)}^2 \leq (nA/\alpha^2) \|\xi\|_{B(2\alpha)}^2$

for any $\xi \in \Lambda^s(M)$, where

$$\|\xi\|_{B(2\alpha)}^2 = \langle\langle \xi, \xi \rangle\rangle_{B(2\alpha)} = \int_{B(2\alpha)} \langle \xi, \xi \rangle * 1.$$

We remark that, for $\xi \in L_2^s(M) \cap \Lambda^s(M)$, $w_\alpha \xi$ belongs in $\Lambda_0^s(M)$ and $w_\alpha \xi \rightarrow \xi$ ($\alpha \rightarrow \infty$) in the strong sense.

For any $\xi \in L_2^1(M) \cap \Lambda^1(M)$, we have

$$(3.1) \quad d\xi_\alpha = w_\alpha^2 d\xi + 2w_\alpha dw_\alpha \wedge \xi,$$

$$(3.2) \quad \delta\xi_\alpha = w_\alpha^2 \delta\xi - *(2w_\alpha dw_\alpha \wedge *\xi),$$

where $\xi_\alpha = w_\alpha^2 \xi$.

LEMMA 2 (cf. [2]). For any $\xi \in L_2^1(M) \cap \Lambda^1(M)$,

$$\langle\langle 2w_\alpha dw_\alpha \wedge \xi, \nabla\xi \rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha \nabla^2\xi, w_\alpha \xi \rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha \nabla\xi, w_\alpha \nabla\xi \rangle\rangle_{B(2\alpha)} = 0,$$

where $(\nabla^2\xi)_i = \nabla^j \nabla_j \xi_i$ and $(\nabla\xi)_{ij} = \nabla_i \xi_j$.

PROOF. For given ξ , we consider a 1-form η defined by

$$\eta = \frac{1}{2} d(\langle\xi, \xi\rangle) = (\nabla_i \xi_j) \xi^j dx^i.$$

Then, $*(w_\alpha^2 \eta)$ being a $(n - 1)$ -form with compact support in $B(2\alpha)$, we have

$$\int_M d(*(w_\alpha^2 \eta)) = 0.$$

On the other hand, we have

$$d(*(w_\alpha^2 \eta)) = -*\delta(w_\alpha^2 \eta).$$

Thus we have

$$\int_M *\delta(w_\alpha^2 \eta) = 0.$$

By (2.2) and (3.2), we have

$$\delta(w_\alpha^2 \eta) = -w_\alpha^2 (\nabla^i \nabla_i \xi_j) \xi^j - w_\alpha^2 (\nabla_i \xi_j) (\nabla^i \xi^j) - *(2w_\alpha dw_\alpha \wedge *\eta).$$

Therefore we have the assertion.

Let ξ be a Killing vector field on M whose dual 1-form with respect to g belongs in $L_2^1(M) \cap \Lambda^1(M)$. By the definition of Δ , (2.2) and (2.5), we have

$$(3.3) \quad \langle\langle \Delta\xi, w_\alpha^2 \xi \rangle\rangle_{B(2\alpha)} - \langle\langle \delta d\xi, w_\alpha^2 \xi \rangle\rangle_{B(2\alpha)} = 0.$$

From (2.3), we have

$$\langle\langle \Delta\xi, w_\alpha^2 \xi \rangle\rangle_{B(2\alpha)} = -\langle\langle w_\alpha \nabla^2\xi, w_\alpha \xi \rangle\rangle_{B(2\alpha)} + \langle\langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle\rangle_{B(2\alpha)},$$

where \mathcal{R} denotes the Ricci transformation on 1-forms defined by $(\mathcal{R}\xi)_i = -R^h_{\ i} \xi_h$.

On the other hand, by (3.1), we have

$$\langle\langle \delta d\xi, w_\alpha^2 \xi \rangle\rangle_{B(2\alpha)} = \langle\langle w_\alpha d\xi, w_\alpha d\xi \rangle\rangle_{B(2\alpha)} + \langle\langle d\xi, 2w_\alpha dw_\alpha \wedge \xi \rangle\rangle_{B(2\alpha)}.$$

By (2.4),

$$\begin{aligned} \langle d\xi, d\xi \rangle &= (1/2!) \{ 2(\nabla_i \xi_k) (\nabla^i \xi^k) - 2(\nabla_i \xi_k) (\nabla^k \xi^i) \} \\ &= (1/2!) \{ 2(\nabla_i \xi_k) (\nabla^i \xi^k) + 2(\nabla_i \xi_k) (\nabla^i \xi^k) \} \\ &= (1/2!) 4(\nabla_i \xi_k) (\nabla^i \xi^k) \\ &= 4\langle \nabla\xi, \nabla\xi \rangle \end{aligned}$$

and we have

$$\|w_\alpha d\xi\|_{B(2\alpha)}^2 = 4\|w_\alpha \nabla\xi\|_{B(2\alpha)}^2.$$

By the Schwarz inequality, Lemma 1 and the above fact, we have

$$\begin{aligned} |\langle \langle d\xi, 2w_\alpha dw_\alpha \wedge \xi \rangle \rangle_{B(2\alpha)}| &\leq \|w_\alpha d\xi\|_{B(2\alpha)} \|2dw_\alpha \wedge \xi\|_{B(2\alpha)} \\ &\leq \frac{1}{2} (\|w_\alpha d\xi\|_{B(2\alpha)}^2 + 4\|dw_\alpha \wedge \xi\|_{B(2\alpha)}^2) \\ &\leq \frac{1}{2} (4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 + (4nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2). \end{aligned}$$

Thus we have, from (3.3),

$$\begin{aligned} \langle \langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} &= \langle \langle w_\alpha \nabla^2 \xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} + \langle \langle w_\alpha d\xi, w_\alpha d\xi \rangle \rangle_{B(2\alpha)} \\ &\quad + \langle \langle d\xi, 2w_\alpha dw_\alpha \wedge \xi \rangle \rangle_{B(2\alpha)} \\ &\geq \langle \langle w_\alpha \nabla^2 \xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} + 4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 \\ &\quad - \frac{1}{2} (4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 + (4nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2) \end{aligned}$$

(by Lemma 2)

$$\begin{aligned} &= -\langle \langle w_\alpha \nabla \xi, w_\alpha \nabla \xi \rangle \rangle_{B(2\alpha)} - \langle \langle 2w_\alpha dw_\alpha \wedge \xi, \nabla \xi \rangle \rangle_{B(2\alpha)} \\ &\quad + 4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 - \frac{1}{2} (4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 + (4nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2) \end{aligned}$$

(by the Schwarz inequality and Lemma 1)

$$\begin{aligned} &\geq -\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 - \frac{1}{2} (\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 + (4nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2) \\ &\quad + 4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 - \frac{1}{2} (4\|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 + (4nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2). \end{aligned}$$

Therefore we have

$$\langle \langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} \geq \frac{1}{2} \|w_\alpha \nabla \xi\|_{B(2\alpha)}^2 - (4nA/\alpha^2)\|\xi\|_{B(2\alpha)}^2.$$

Letting $\alpha \rightarrow \infty$, we have

LEMMA 3. *Let ξ be a Killing vector field on M with finite global norm. If $\limsup_{\alpha \rightarrow \infty} \langle \langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} < \infty$, then*

$$\limsup_{\alpha \rightarrow \infty} \langle \langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} \geq \frac{1}{2} \|\nabla \xi\|^2.$$

THEOREM 1. *If M is a complete Riemannian manifold with nonpositive Ricci curvature, then every Killing vector field on M with finite global norm is a parallel vector field.*

PROOF. By the nonpositivity of Ricci curvature, we have

$$\limsup_{\alpha \rightarrow \infty} \langle \langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle \rangle_{B(2\alpha)} \leq 0$$

for any Killing vector field ξ on M with finite global norm. From Lemma 3, we have $\nabla \xi = 0$.

Since the length of a parallel vector field is constant, we have

COROLLARY 1. *Let M be a complete noncompact Riemannian manifold with nonpositive Ricci curvature. If there exists a nontrivial Killing vector field on M with finite global norm, then the volume of M is finite.*

The following example illustrates the role of the hypothesis that M has nonpositive Ricci curvature in the above results.

EXAMPLE 1. We take four real numbers a_1, a_2, a_3 and a_4 such that $0 < a_1 < a_2 < a_3 < a_4 < 1$ and fix them. We consider two C^∞ -functions $h_1, h_2: (0, \infty) \rightarrow \mathbf{R}$ satisfying $0 \leq h_i(r) \leq 1$ ($i = 1, 2$) for $0 < r$ and

$$\begin{aligned} h_1(r) &= 1, & h_2(r) &= 0 & \text{for } 0 < r < a_2, \\ h_1(r) &= 0, & h_2(r) &= 1 & \text{for } a_3 \leq r. \end{aligned}$$

We define functions f_i, g_i ($i = 1, 2$) as follows; $f_1(r) = h_1(r)r^{-2}(\log r)^{-2}$, $g_1(r) = h_1(r)(\log r)^{-2}$ for $0 < r < a_4$ and $f_2(r) = h_2(r)$, $g_2(r) = h_2(r)r^{-4/3}$ for $a_1 < r$. Then we set

$$\begin{aligned} F_1(r) &= f_1(r) & (0 < r \leq a_3), & & = 0 & (a_3 < r), \\ F_2(r) &= 0 & (0 < r < a_2), & & = f_2(r) & (a_2 \leq r), \\ G_1(r) &= g_1(r) & (0 < r \leq a_3), & & = 0 & (a_3 < r), \\ G_2(r) &= 0 & (0 < r < a_2), & & = g_2(r) & (a_2 \leq r), \end{aligned}$$

and

$$F(r) = F_1(r) + F_2(r), \quad G(r) = G_1(r) + G_2(r) \quad \text{for } 0 < r.$$

The functions F and G are C^∞ and $F(r), G(r) > 0$ for $0 < r$.

Let $M = R^2 - \{(0, 0)\} = \{(r, \theta) | 0 < r, 0 \leq \theta < 2\pi\}$ and $ds^2 = F(r)(dr)^2 + G(r)(d\theta)^2$. Then (M, ds^2) is a complete Riemannian manifold. A vector field $\xi = \partial/\partial\theta$ on M is a Killing vector field with respect to the Riemannian metric ds^2 . Since $\int_0^{a_2} r^{-1}(\log r)^{-N} dr < \infty$ ($N = 2, 3, \dots$), $\int_{a_3}^\infty r^{-2/3} dr = \infty$ and $\int_{a_3}^\infty r^{-L} dr < \infty$ ($1 < L$), we have that the volume of M is infinite, $\|\xi\|$ is finite and $0 < \langle\langle \mathcal{R}\xi, \xi \rangle\rangle < \infty$.

REMARK TO COROLLARY 1. Every complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinite volume (cf. S.-T. Yau [Indiana Univ. Math. J. 25 (1976), 659–670] and E. Calabi [Notices Amer. Math. Soc. 22 (1975), A205]).

THEOREM 2. *If M is a complete Riemannian manifold with negative Ricci curvature, then there is no nontrivial Killing vector field on M with finite global norm.*

PROOF. Let ξ be a Killing vector field on M with finite global norm. By the negativity of Ricci curvature, we have $\langle\langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle\rangle_{B(2\alpha)} < 0$ for every α . From Lemma 3,

$$0 \geq \limsup_{\alpha \rightarrow \infty} \langle\langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle\rangle_{B(2\alpha)} \geq \frac{1}{2} \|\nabla \xi\|^2 \geq 0.$$

Thus we have $\langle\langle w_\alpha \mathcal{R}\xi, w_\alpha \xi \rangle\rangle_{B(2\alpha)} = 0$ for every α . By the negativity of Ricci curvature, $\xi = 0$.

REMARK. There is a similar discussion for holomorphic vector fields on complete Kähler manifolds with finite global norms [5].

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REFERENCES

1. A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 313–362.
2. H. Kitahara, *Non-existence of non-trivial \square' -harmonic 1-forms on a complete foliated riemannian manifold*, Trans. Amer. Math. Soc. **262** (1980), 429–435.
3. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.
4. G. de Rham, *Variétés différentiables*, Hermann, Paris, 1955.
5. S. YoroZu, *Holomorphic vector fields on complete Kähler manifolds*, Ann. Sci. Kanazawa Univ. **17** (1980), 17–21.

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