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ON THE EXPLICIT FORM OF THE DENSITY OF BROWNIAN EXCURSION LOCAL TIME

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ABSTRACT. Let $\underline{W}_0^+(t)$, $0 \leqslant t \leqslant 1$, denote Brownian excursion and let $\underline{l}_0^+(v)$, $v > 0$, be its local time at level v. Starting from a representation of the density of $\underline{l}_0^+(v)$ as a complex integral we derive an explicit form of this density, written as an infinite series involving the n-fold convolution of known densities. Finally the result is used as an alternative check of Knight's result on the same topic.

1. Introduction. In [**2**] and [**4**] an expression is given for the distribution of Brownian excursion local time $\underline{l}_0^+(v)$, $v > 0$. In these papers this distribution is represented as a complex integral and it is easily verified that this representation is equivalent to the double Laplace transform given in Formula (3.17) of [**3**] (for this verification see §2). Although both relations seem difficult to invert (cf. [**3**, final part of §3]) we have succeeded in giving an explicit form for the density of $\underline{l}_0^+(v)$, $v > 0$, as an infinite series involving the n-fold convolution of known densities. In the final part of this note it is shown that our result agrees with that of Knight (cf. [**5**]).

To read this note no knowledge of complex integration other than the inversion theorem for Laplace transforms is necessary.

2. The explicit form. Denote by S the path (straight line) from $a - i\infty$ to $a + i\infty$, $a > 0$. Then according to [**2**, Formula (5.9)] or [**4**, Formula VI (3.36)] for $v > 0$ and $y \geqslant 0$,

$$P\{\underline{l}_0^+(v) \leqslant y\} = 1 - \frac{1}{2i\sqrt{2\pi}} \int_S \sqrt{z}\, e^{z/2} \frac{e^{-v\sqrt{z}}}{\sinh\{v\sqrt{z}\}} \exp\left\{ -\frac{y}{2} \frac{\sqrt{z}\, e^{v\sqrt{z}}}{\sinh\{v\sqrt{z}\}} \right\} dz, \tag{2.1}$$

where $\underline{l}_0^+(v)$ is the local time at level $v > 0$ of Brownian excursion. To obtain the equivalence of (2.1) with Formula (3.17) of [3], observe that if $\underline{z}(\cdot, t)$ is the local time of the excursion straddling t, and $\underline{l}(t)$ its length, then

$$\underline{z}(v, t) \stackrel{d}{=} (\underline{l}(t))^{1/2} \underline{l}_0^+\left(v(\underline{l}(t))^{-1/2}\right), \tag{2.2}$$

with $\underline{l}_0^+(v)$ and $\underline{l}(t)$ independent.

Since the distribution of $\underline{l}(t)$ is known (cf. [**1**, (4.4)]), it is easy to obtain Formula (3.17) of [**3**] from Relation (2.1). W. Vervaat (private communication) has pointed

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out that the converse statement is also true, because it follows from Relation (2.2) and Formula (3.17) of [3] that for $\mu, \rho \geqslant 0$,

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-ux}}{x^{3/2}}\left(1 - E\exp\left(-\rho\sqrt{x}\, \underline{l}_0^+(x^{-1/2})\right)\right) dx = \frac{2\rho\sqrt{2u}\, e^{-2\sqrt{2u}}}{\sqrt{2u} + \rho\left(1 - e^{-2\sqrt{2u}}\right)}.$$

The latter relation uniquely determines Relation (2.1).

We shall derive an explicit expression for the density of $\underline{l}_0^+(v)$. Observe from (2.1) that the density of $\underline{l}_0^+$ at level v is given for $y > 0$ by

$$\text{(2.3)} \qquad \frac{1}{4i\sqrt{2\pi}}\int_S e^{z/2}\frac{z}{\sinh^2\{v\sqrt{z}\,\}} \exp\left\{-\frac{y}{2}\frac{\sqrt{z}\, e^{v\sqrt{z}}}{\sinh\{v\sqrt{z}\,\}}\right\} dz.$$

Now use the fact that

$$\frac{\sqrt{z}\, e^{v\sqrt{z}}}{2\sinh\{v\sqrt{z}\,\}} = \sqrt{z} + \frac{\sqrt{z}\, e^{-v\sqrt{z}}}{2\sinh\{v\sqrt{z}\,\}}$$

and the series expansion of $\exp(\cdot)$ to obtain, for $y, v > 0$,

(2.4)

$$\frac{d}{dy}P\{\underline{l}_0^+(v) \leqslant y\}$$

$$= (4iv^2\sqrt{2\pi}\,)^{-1}\sum_{n=0}^\infty \frac{(-y/2v)^n}{n!}\int_S e^{z/2}\frac{v^2 z}{\sinh^2\{v\sqrt{z}\,\}}\left[\frac{v\sqrt{z}\, e^{-v\sqrt{z}}}{\sinh\{v\sqrt{z}\,\}}\right]^n e^{-y\sqrt{z}}\, dz.$$

We define $x := y/v$, $\lambda := v^2 z/\pi^2$. From this substitution it follows that the r.h.s. of (2.4) equals

(2.5)

$$\frac{\pi^2}{4iv^4\sqrt{2\pi}}\sum_{n=0}^\infty \frac{(-x/2)^n}{n!}\int_S e^{\lambda(\pi^2/2v^2)}\frac{4\pi^2\lambda e^{-2\pi\sqrt{\lambda}}}{\left(1 - e^{-2\pi\sqrt{\lambda}}\right)^2}\left[\frac{2\pi\sqrt{\lambda}\, e^{-2\pi\sqrt{\lambda}}}{1 - e^{-2\pi\sqrt{\lambda}}}\right]^n e^{-x\pi\sqrt{\lambda}}\, d\lambda.$$

It is known from [**1**, p. 168] that

$$\text{(2.6)} \qquad \int_0^\infty e^{-\lambda s}\, dF_1(s) = \frac{2\pi\sqrt{\lambda}\, e^{-\pi\sqrt{\lambda}}}{\left(1 - e^{-2\pi\sqrt{\lambda}}\right)},$$

$$\text{(2.7)} \qquad \int_0^\infty e^{-\lambda s}\, dF_2(s) = \frac{4\pi^2\lambda e^{-2\pi\sqrt{\lambda}}}{\left(1 - e^{-2\pi\sqrt{\lambda}}\right)^2},$$

where F_1 and F_2 are proper distribution functions ($F_1(2s^2)$, $F_2(2s^2)$ is the distribution function of the maximum of Brownian meander, the maximum of Brownian excursion, respectively) defined by

$$\text{(2.8)} \qquad F_1(s) := \sum_{n=-\infty}^\infty (-1)^n e^{-n^2 s}, \quad s > 0,$$

$$\text{(2.9)} \qquad F_2(s) := 1 + 2\sum_{n=1}^\infty e^{-n^2 s}(1 - 2n^2 s), \quad s > 0.$$

Note that if f_i is the density of F_i, then $f_2 = f_1^{2*}$ (cf. [1] or [5]). Further it is well known that for $k > 0$

$$\int_0^\infty e^{-\lambda s} \frac{k}{2\sqrt{\pi s^3}} e^{-k^2/4s}\, ds = e^{-k\sqrt{\lambda}}. \tag{2.10}$$

Hence, it follows from (2.4)–(2.10) and the Laplace inversion theorem that for $y, v > 0$

$$\frac{d}{dy} P\{\underline{l}_0^+(v) \leqslant y\} = 2^{-3/2}\pi^{5/2}v^{-4} \sum_{n=0}^{\infty} \frac{(-y)^n}{n! 2^n v^n} g_x * f_2 * \{g_1 * f_1\}^{n*}\left(\frac{\pi^2}{2v^2}\right), \tag{2.11}$$

where the densities f_i and g_x on $(0, \infty)$ are defined by

$$f_i(u) = F_i'(u), \quad i = 1, 2, \qquad g_x(u) = \frac{x\sqrt{\pi}}{2u^{3/2}} e^{-x^2\pi^2/4u}, \quad x > 0.$$

Recently F. B. Knight obtained in an entirely different way an explicit form for the density of $\underline{l}_0^+(v)$. Knight's result (cf. [5]) states that, for $y, v > 0$,

$$\frac{d}{dy} P\{\underline{l}_0^+(v) \leqslant y\} = 2^{-1/2}\pi^{5/2}v^{-3} \int_0^1 f_2\left(\frac{\pi^2}{2v^2}(1-t)\right) f(t, y)\, dt,^1 \tag{2.12}$$

where the density f_2 is defined above and

$$f(t, y) := -\frac{(2\pi t)^{-1/2}}{2v} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} \left\{ y^n \frac{d^2}{dy^2} e^{-2v^2(y+n)^2/t} \right\}. \tag{2.13}$$

A direct verification (without Laplace transforms) of the equivalence of (2.11) and (2.12) seems too difficult. For this reason we follow the final remark of Knight in [5], which states that the r.h.s. of (2.12) is a convolution and that passing to Laplace transforms might give a simplification of (2.12). Knight states that the Laplace transform then obtained seems too difficult for inversion but in fact it turns out that this Laplace transform is equivalent with (2.3); consequently (2.11) and (2.12) give the same expression and the desired simplification of (2.12) is relation (2.11). We shall prove this in the following.

Observe that the r.h.s. of (2.12) can be written as

$$2^{1/2}\pi^{1/2}v^{-1} \int_0^{\pi^2/2v^2} f_2\left(\frac{\pi^2}{2v^2} - u\right) f\left(\frac{2v^2}{\pi^2}u, y\right) du. \tag{2.14}$$

The integral is a convolution integral. The joint Laplace transform of $f(t, y)$ in the variables λ and β is given by (cf. [5, Lemma 2.2]),

$$(2v)^{-1} \frac{1 - e^{-2v\sqrt{2\lambda}}}{\sqrt{2\lambda} + \beta\left(1 - e^{-2v\sqrt{2\lambda}}\right)}.$$

[1]As remarked by the author himself, the density in Theorem 2.3 of [5] is a factor four too large due to an oversight in Theorem 1.1 of [5] (cf. [6]).

The Laplace transform in β is a simple Laplace transform of the negative exponential distribution, hence

$$\int_0^\infty e^{-\lambda t} f(t,y)\,dt = (2v)^{-1}\exp\left\{-\frac{y}{2}\frac{\sqrt{2\lambda}\,e^{v\sqrt{2\lambda}}}{\sinh\{v\sqrt{2\lambda}\}}\right\}. \tag{2.15}$$

The relations (2.7) and (2.15) imply that

$$f_2(t) = (2\pi i)^{-1}\int_S e^{\lambda t}\frac{\pi^2\lambda}{\sinh^2\{\pi\sqrt{\lambda}\}}\,d\lambda, \tag{2.16}$$

$$f(t,y) = (2\pi i)^{-1}\int_S e^{\lambda t}(2v)^{-1}\exp\left\{-\frac{y}{2}\frac{\sqrt{2\lambda}\,e^{v\sqrt{2\lambda}}}{\sinh\{v\sqrt{2\lambda}\}}\right\}d\lambda. \tag{2.17}$$

It follows from (2.14)–(2.17) and the substitution $z = \pi^2\lambda/v^2$ that the density given by Knight's formula is equal to

$$(4i\sqrt{2\pi})^{-1}\int_S e^{z/2}\frac{z}{\sinh^2\{v\sqrt{z}\}}\exp\left\{-\frac{y}{2}\frac{\sqrt{z}\,e^{v\sqrt{z}}}{\sinh\{v\sqrt{z}\}}\right\}dz. \tag{2.18}$$

Indeed the latter expression is equal to the one in Formula (2.3).

REMARK. Integration of (2.3) over y from 0 to ∞ leads to

$$(i\sqrt{2\pi})^{-1}\int_S e^{z/2}\frac{\sqrt{z}\,e^{-2v\sqrt{z}}}{1-e^{-2v\sqrt{z}}}\,dz. \tag{2.19}$$

This expression has to be equal to $1 - F_2(2v^2)$, since the density of the local time has an atom of magnitude $F_2(2v^2)$ at 0 ($F_2(2v^2)$ is the probability that the maximum of Brownian excursion is smaller than v). Indeed this equality is easily verified by expanding the numerator of the integrand in (2.19) in powers of $\exp\{-2v\sqrt{z}\}$ and by using Relation (2.10).

Because it is also readily verified from (2.6) and (2.10) that Expression (2.19) equals $(\pi^2\sqrt{2\pi}/2v^3)(f_1 * g_1)(\pi^2/2v^2)$, we have as by-product the relation

$$(f_1 * g_1)\left(\frac{\pi^2}{2v^2}\right) = \frac{2v^3}{\pi^2\sqrt{2\pi}}(1 - F_2(2v^2)). \tag{2.20}$$

Relation (2.20) can be used to give an alternative form of (2.11).

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