PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

POLYNOMIAL GENERATORS FOR $H_*(BSU)$ AND $H_*(BSO; Z_2)$

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ABSTRACT. Specific formulas are given for choosing polynomial generators of $H_*(BSU; R)$, for various R, in terms of the canonical polynomial generators of $H_*(BU; R)$. The analogous formulas for polynomial generators of $H_*(BSO; Z_2)$ are also given.

1. Introduction. Let R be a commutative ring. Then $H^*(BU; R) = R[C_1, \ldots, C_n, \ldots]$ where $C_n \in H^{2n}(BU; R)$ is the nth Chern class with coproduct $\Delta(C_n) = \sum_{k=0}^n C_k \otimes C_{n-k}$. See [**2**, 9–39]. Let $a_n = (C_1^n)^*$ in the dual basis of the basis of $H^*(BU; R)$ of monomials in the Chern classes. Then $H_*(BU; R) = R[a_1, \ldots, a_n, \ldots]$ with coproduct $\Delta(a_n) = \sum_{i=0}^n a_i \otimes a_{n-i}$. Let f: $BSU \to BU$ be the canonical map. Then f^*: $H^*(BU; R) \to H^*(BSU; R)$ is the canonical projection map from $R[C_1, \ldots, C_n, \ldots]$ to $R[C_1, \ldots, C_n, \ldots]/(C_1)$. Dually f_*: $H_*(BSU; R) \to H_*(BU; R)$ is a monomorphism and $H_*(BSU; R) = R[Y_2, \ldots, Y_n, \ldots]$ with deg $Y_n = 2n$ [**1**, Lemma 2.4]. In this paper we define specific polynomial generators $Y_2, \ldots, Y_n, \ldots$ which have simple explicit expressions as polynomials in the $a_1, \ldots, a_n, \ldots$.

In §2 we define a coaction ψ on $H_*(BU; R)$ such that Image f_* equals the elements of $H_*(BU; R)$ which are primitive under the coaction ψ. We apply [**4**, Theorem 2.1] in §3 to define polynomial generators for $H_*(BSU; Q)$. We then determine polynomial generators for $H_*(BSU; Z_{(p)})$ and $H_*(BSU; Z_p)$ with p prime in §4. We give the corresponding results for $H_*(BSO; Z_2)$ in §5.

2. A coaction on $H_*(BU; R)$. The cup-product on $H^*(BU; R)$ defines a module structure ϕ: $R[C_1] \otimes H^*(BU; R) \to H^*(BU; R)$. Dually we have a coaction

$$\psi: H_*(BU; R) \to \Gamma \otimes H_*(BU; R).$$

$\Gamma = \bigoplus_{n=0}^\infty R\gamma_n$ is the divided polynomial Hopf algebra with deg $\gamma_n = 2n$, $\gamma_m\gamma_n = (m, n)\gamma_{m+n}$ and $\psi(\gamma_n) = \sum_{k=0}^n \gamma_k \otimes \gamma_{n-k}$. We collect the basic properties of ψ in the following theorem.

THEOREM 2.1. *ψ: $H_*(BU; R) \to \Gamma \otimes H_*(BU; R)$ is a coassociative counital coaction and an algebra homomorphism. The primitive elements of $H_*(BU; R)$ under ψ are*

$$P_\psi H_*(BU; R) = \text{Image} f_* \cong H_*(BSU; R).$$

Received by the editors January 12, 1981.

1980 *Mathematics Subject Classification.* Primary 55R45; Secondary 57T05.

Key words and phrases. Polynomial generator, classifying space, classical group, coaction, primitive, Chern class, Stiefel-Whitney class.

[1]This research was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

PROOF. ϕ is unital and associative, so $\psi = \phi^*$ is counital and coassociative. Let $Y \in H^*(BU; R)$ with $\Delta(Y) = \Sigma_j Y_j' \otimes Y_j''$. Then

$$\Delta\phi(C_1^n \otimes Y) = \Delta(C_1^n Y) = \sum_{i=0}^{n} \sum_j (i, n-i) C_1^i Y_j' \otimes C_1^{n-i} Y_j''$$

$$= \sum_{i=0}^{n} \sum_j (i, n-i) \phi\left(C_1^i \otimes Y_j'\right) \otimes \phi\left(C_1^{n-i} \otimes Y_j''\right)$$

$$= (\phi \otimes \phi) \circ (1 \otimes T \otimes 1) \circ (\Delta \otimes \Delta)(C_1^n \otimes Y).$$

Thus ϕ is a map of coalgebras, so $\psi = \phi^*$ is an algebra homomorphism. Alternatively ψ is induced by ψ': $U \xrightarrow{\Delta} U \times U \xrightarrow{\det \times 1} U(1) \times U$, and chasing the relevant diagrams for ψ' shows ψ to be coassociative, counital and an algebra homomorphism. If $Z \in H_*(BSU; R)$, $Y \in H^*(BU; R)$ and $s > 0$ then $\langle \psi f_*(Z), C_1^s \otimes Y \rangle = \langle f_*(Z), C_1^s Y \rangle = \langle Z, f^*(C_1^s Y) \rangle = 0$. Thus Image $f_* \subset P_\psi H_*(BU; R)$. If $X \in P_\psi H_*(BU; R)$ and $s > 0$ then $\langle X, C_1^s Y \rangle = \langle \psi(X), C_1^s \otimes Y \rangle = 0$. Thus X is an R-linear combination of the $(C_{n_1} \ldots C_{n_t})^*$ with $2 \leqslant n_1 \leqslant \cdots \leqslant n_t$. These $(C_{n_1} \ldots C_{n_t})^*$ are an R-basis for Image f_*. Q.E.D.

When $R = Z_p$, p prime, all pth powers of positive degree elements are zero in Γ. Thus all pth powers in $H_*(BU; Z_p)$ are ψ-primitive. We thus have the following consequence of Theorem 2.1.

COROLLARY 2.2. $\{x^p | x \in H_*(BU; Z_p)\} \subset$ Image f_*.

3. Polynomial generators for $H_*(BSU; Q)$. Throughout this section let p be a fixed prime. We will define a sequence of elements $G_{p,k}$ in $H_*(BSU)$ which are polynomial generators for $H_*(BSU; Q)$. We wish to apply [4, Theorem 2.1] to a subcomodule $\{Y_2, \ldots, Y_n, \ldots\}$ of $H_*(BU)$ such that $\psi(Y_n) \subset \Sigma_{i=0}^{n-1} \Gamma_{2i} \otimes H_{2n-2i}(BU)$. Since $\psi(a_n) = \Sigma_{i=0}^{n} \gamma_i \otimes a_{n-i}$ contains $\gamma_n \otimes 1$ as a summand we cannot let Y_n equal a_n. We construct suitable Y_n in the following lemma.

LEMMA 3.1. *Let $Y_{p,n} = p^n a_n - \Sigma a_{i_1} \ldots a_{i_p}$ for $n \geqslant 1$ where the sum is taken over all $(i_1, \ldots, i_p)$ such that the $i_k \geqslant 0$ and $i_1 + \cdots + i_p = n$. Then $Y_{p,1} = 0$ and*

$$\psi(Y_{p,n}) = \sum_{k=0}^{n-2} p^k \gamma_k \otimes Y_{p,n-k}.$$

PROOF. $Y_{p,1} = pa_1 - \Sigma 1 \ldots 1 a_1 1 \ldots 1 = pa_1 - pa_1 = 0$. Let $n \geqslant 2$. The summand of $\psi(Y_{p,n})$ in $\Gamma_{2n} \otimes H_0(BU)$ is $p^n\gamma_n \otimes 1 - \Sigma \gamma_{i_1} \ldots \gamma_{i_p} \otimes 1 = p^n \gamma_n \otimes 1 - \Sigma(i_1, \ldots, i_p)\gamma_n \otimes 1 = p^n\gamma_n \otimes 1 - (1 + \cdots + 1)^n \gamma_n \otimes 1 = p^n\gamma_n \otimes 1 - p^n\gamma_n \otimes 1 = 0$. The summand of $\psi(Y_{p,n})$ in $\Gamma_{2k} \otimes H_{2n-2k}(BU)$ for $k > 0$ is $y = p^n\gamma_k \otimes a_{n-k} - \Sigma \gamma_{i_1} \ldots \gamma_{i_p} \otimes a_{h_1} \ldots a_{h_p}$ where the sum is taken over all $(i_1, \ldots, i_p)$ and $(h_1, \ldots, h_p)$ with $h_t \geqslant 0$, $i_t \geqslant 0$, $i_1 + \cdots + i_p = k$, $h_1 + \cdots + h_p = n - k$. Thus

$$y = p^n\gamma_n \otimes a_{n-k} - \Sigma(i_1, \ldots, i_p)\gamma_k \otimes a_{h_1} \ldots a_{h_p}$$

$$= p^n\gamma_k \otimes a_{n-k} - \Sigma(1 + \cdots + 1)^k \gamma_k \otimes a_{h_1} \ldots a_{h_p}$$

$$= p^k\gamma_k \otimes \left[p^{n-k}a_{n-k} - \Sigma\gamma_k \otimes a_{h_1} \ldots a_{h_p}\right] = p^k\gamma_k \otimes Y_{p,n-k}.$$

Note that y is zero when $k = n - 1$. Q.E.D.

We now apply [4, Theorem 2.1] to the $Y_{p,2}, \dots, Y_{p,n}, \dots$.

THEOREM 3.2. *Let p be a prime. Define $G_{p,n} \in H_{2n}(BU)$ inductively on $n \geqslant 2$ by*

$$G_{p,n} = Y_{p,n} - \sum_{k=2}^{n-1} p^{n-k} a_{n-k} G_{p,k}.$$

Then

(a) $G_{p,n} = Y_{p,n} + \sum_{k=2}^{n-1} p^{n-k} \chi(a_{n-k}) Y_{n-k}$ *and*

$$\chi(a_t) = \sum_{e_1 + 2e_2 + \cdots + te_t = t} (-1)^{e_1 + \cdots + e_t} (e_1, \dots, e_t) a_1^{e_1} \dots a_t^{e_t};$$

(b) $H_*(BU; Q) = Q[a_1, G_{p,2}, \dots, G_{p,n}, \dots]$;

(c) $H_*(BSU; Q) \cong \operatorname{Image} f_* = Q[G_{p,2}, \dots, G_{p,n}, \dots]$.

PROOF. We apply [4, Theorem 2.1] to the algebra $H_*(BU; Q)$ with its polynomial generators $G = \{pa_1, Y_{p,2}, \dots, Y_{p,n}, \dots\}$. $H_*(BU; Q)$ has the coaction ψ and by Lemma 3.1 the Q-space with basis G is a subcomodule of $H_*(BU; Q)$. In the notation of [4, Theorem 2.1] we have $\theta_{n,k} = p^{n-k}\gamma_{n-k}$ for $2 \leqslant k \leqslant n$, $\theta_{n,1} = \theta_{n,0} = 0$ for $n \geqslant 2$, $\theta_{1,1} = 1$ and $\theta_{1,0} = p\gamma_1$. Define $\phi_{n,k} = p^{n-k}a_{n-k}$ for $0 \leqslant k \leqslant n$. Then $\psi(\phi_{n,k}) = \sum_{i=0}^{n-k} p^i \gamma_i \otimes p^{n-k-i} a_{n-k-i} = \sum_{i=0}^{n-k} \theta_{n,n-i} \otimes \phi_{n-i,k} = \sum_{h=k}^{n} \theta_{n,h} \otimes \phi_{h,k}$ where $h = n - i$. Let $S = \{2, 3, \dots\}$ since $\theta_{n,1} = \theta_{n,0} = 0$ for $n \geqslant 2$. Thus the hypotheses of [4, Theorem 2.1] are satisfied. We therefore conclude the first part of (a), (b) and $P_\psi H_*(BU; Q) = Q[G_{p,2}, \dots, G_{p,n}, \dots]$. Now (c) follows from Theorem 2.1. Observe that the coproduct which [4, Theorem 2.1] defines on $H_*(BU)$ is the canonical one. Thus the second part of (a) follows from [4, Corollary 4.2(v)]. Q.E.D.

Observe that $G_{p,n} = pG'_{p,n}$ in $H_*(BSU)$ if $n \not\equiv 0 \bmod p$. The following criterion shows that the only $G_{p,n}$ or $G'_{p,n}$ which is a polynomial generator of $H_*(BSU)$ is $G_{2,2}$.

THEOREM 3.3. *Let $\nu(n)$ be p if $n = p^s$, p prime, and let $\nu(n)$ be 1 if n is not a power of a prime. Then an element G of $H_{2n}(BSU)$ is a polynomial generator if and only if $f_*(G) \equiv \pm \nu(n) a_n$ modulo decomposables.*

PROOF. Let $PH^{2n}(BU) = Zp_n$. Write $n = q_1^{s_1} \dots q_t^{s_t}$ with $q_1, \dots, q_t$ distinct primes. Then $f^*(p_n)$ contains $\sum_{i=1}^{t} \pm q_1^{s_1} \dots \widehat{q_i^{s_i}} \dots q_t^{s_t} C^{q_i^{s_i}}_{q_1^{s_1} \dots q_i^{s_i} \dots \widehat{q_i^{s_i}}}$ as a summand. Thus $PH^{2n}(BSU) = Zf^*(p_n)$ when $t \geqslant 2$. Observe that $f^*(p_{q^s})$ contains $\pm qC^{q}_{q^s-1}$ as a summand. In addition q divides $f^*(p_{q^s})$ because $f^*(p_{q^s}) \equiv f^*(C_1^{q^s}) \equiv 0 \bmod q$. Thus $PH^{2q^s}(BSU) = Z[f^*(p_{q^s})\frac{1}{q}]$, and $PH^{2n}(BSU) = Z[f^*(p_n)\frac{1}{\nu(n)}]$ for all $n > 2$.

Hence $QH_{2n}(BSU) \to QH_{2n}(BU)$ is $Z \overset{\nu(n)}{\rightarrowtail} Z$. Q.E.D.

4. Polynomial generators for $H_*(BSU; Z_{(p)})$ and $H_*(BSU; Z_p)$. We begin with a criterion for determining whether a given element of $H_*(BSU; Z_{(p)})$ or $H_*(BSU; Z_p)$ is a polynomial generator. We then use the elements of $H_*(BSU)$ defined in §3 to construct polynomial generators for $H_*(BSU; Z_{(p)})$ and for $H_*(BSU; Z_p)$. We conclude by noting a simple polynomial generator for $H_{4n}(BSU; Z_p)$, p odd. Let p be a fixed prime throughout this section.

THEOREM 4.1. (a) *G is a polynomial generator of $H_{2n}(BSU; Z_{(p)})$ if and only if $f_*(G) \equiv \mu a_n$ modulo decomposables where*

$$\begin{cases} p \nmid \mu & \text{if } n \text{ is not a power of } p, \\ p^2 \nmid \mu & \text{if } n \text{ is a power of } p. \end{cases}$$

(b) *For n not a power of p, G is a polynomial generator of $H_{2n}(BSU; Z_p)$ if and only if $f_*(G) \equiv \mu a_n$ modulo decomposables with $0 \neq \mu \in Z_p$. When $n = p^s$, $a^p_{p^{s-1}}$ is f_* of a polynomial generator of $H_{2p^s}(BSU; Z_p)$.*

PROOF. Note that (a) follows from (b) and the observation that $f_*(G_{p,p^s}) \equiv (p^{p^s} - p)a_{p^s}$ modulo decomposables. To prove (b) write $PH^{2n}(BU; Z_p) = Z_p\mathcal{P}_n$, $n = p^s m$ with $m \not\equiv 0 \bmod p$. Then $\mathcal{P}_n = \mathcal{P}_m^{p^s}$ so by induction on degree

$$PH^{2n}(BSU; Z_p) = Z_p f^*(\mathcal{P}_{2n})$$

when n is not a power of p. This gives (b) in this case. When $n = p^s$, $PH^{2p^s}(BSU; Z_p) = Z_p f^*[\frac{1}{p}(\mathcal{P}_p - C_1^p)]^{p^{s-1}}$ and

$$\begin{aligned} \left\langle \left[\frac{1}{p}(\mathcal{P}_p - C_1^p)\right]^{p^{s-1}}, a^p_{p^{s-1}} \right\rangle &= \left\langle \left[\frac{1}{p}\Delta^{p-1}(\mathcal{P}_p - C_1^p)\right]^{p^{s-1}}, a_{p^{s-1}} \otimes \cdots \otimes a_{p^{s-1}} \right\rangle \\ &= -[(p-1)!]^{p^{s-1}} \left\langle C_1^{p^{s-1}} \otimes \cdots \otimes C_1^{p^{s-1}}, a_{p^{s-1}} \otimes \cdots \otimes a_{p^{s-1}} \right\rangle \\ &= -[(p-1)!]^{p^{s-1}} \not\equiv 0 \end{aligned}$$

modulo p. Thus by Corollary 2.2, $a^p_{p^{s-1}}$ is f_* of a polynomial generator. Q.E.D.

THEOREM 4.2. *In $H_*(BU; Z_{(p)})$ define G_n' and then V_n' by induction on n from the following formulas:*

$$\begin{aligned} G_n' &= \frac{1}{p} G_{p,n} \quad \text{if } n \not\equiv 0 \bmod p,\ n > 2, \\ G_{p^s}' &= G_{p,p^s}, \\ G_{mp}' &= \frac{1}{p}\left[G_{p,mp} - V_m'\right] \quad \text{if } m > 2,\ m \neq p^s, \\ V_{p^s}' &= G_{p^{s+1}}'. \end{aligned}$$

If $n \neq p^s$, $n > 2$ and $G_n' = (p^{n-1} - 1)a_n + \Sigma\alpha_{e_1,\dots,e_t} a_1^{e_1} \cdots a_t^{e_t}$ with the $e_i > 0$ and $\alpha_{e_1,\dots,e_t} \in Z$ then

$$V_n' = \frac{1}{p^{n-1} - 1}\left[G_n'^p - \sum \alpha_{e_1,\dots,e_t} V_1'^{e_1} \cdots V_t'^{e_t}\right].$$

Then $V_n' \equiv a_n^p$ modulo p and

$$H_*(BSU; Z_{(p)}) \cong \operatorname{Image} f_* = Z_{(p)}[G_2', \dots, G_n', \dots].$$

PROOF. Observe that $G_{p,n}$ is divisible by p if $n \not\equiv 0 \bmod p$. Also $G_{p,pm} \equiv a_m^p$ modulo p. By the induction hypothesis $G_{p,pm} - V_m'$ is divisible by p when m is not a power of p. Thus all the G_n' are well-defined elements of $f_*(H_*(BSU; Z_{(p)}))$. Note that $G'_n \equiv (p^{n-1} - 1)a_n$ modulo decomposables if n is not a power of p. In this case

$$G'^p_n \equiv (p^{n-1} - 1)^p a_n^p + \sum \alpha^p_{e_1, \dots, e_t} a_1^{pe_1} \dots a_t^{pe_t} \bmod p$$
$$\equiv (p^{n-1} - 1)a_n^p + \sum \alpha_{e_1, \dots, e_t} V_1'^{e_1} \dots V_t'^{e_t} \bmod p.$$

Thus $V'_n \equiv a_n^p \bmod p$ when n is not a power of p. $V'_{p^s} = G'_{p^{s+1}} = G_{p,p^{s+1}} \equiv a^p_{p^s} \bmod p$. Also $G'_{p^s} \equiv (p^{p^s} - p)a_{p^s}$ modulo decomposables. By Theorem 4.1 the $G'_2, \dots, G'_n, \dots$ are polynomial generators for Image f_*. Q.E.D.

When working mod p one can replace the $G_{p,n}$ by the simpler $Y_{p,n}$ in most cases.

THEOREM 4.3. *In $H_*(BU; Z_{(p)})$ define the G_n and then V_n by induction on n from the following formulas*:

$$G_n = \frac{1}{p} Y_{p,n} \quad \text{if } n \not\equiv 0, 1 \bmod p, n > 2,$$
$$G_{p^s} = Y_{p,p^s} \quad \text{if } s \geqslant 1,$$
$$G_{mp} = \frac{1}{p}\left[Y_{p,mp} - V_m \right] \quad \text{if } m > 2, m \neq p^s,$$
$$G_{mp+1} = \frac{1}{p} Y_{p,mp+1} - a_1 a_m^p \quad \text{if } m > 1,$$
$$V_{p^s} = G_{p^{s+1}} \quad \text{if } s \geqslant 0.$$

If $n \neq p^s$, $n \geqslant 2$ and $G_n = (p^{n-1} - 1)a_n + \Sigma\alpha_{e_1, \dots, e_t} a_1^{e_1} \dots a_t^{e_t}$ with the $e_i > 0$ and $\alpha_{e_1, \dots, e_t} \in Z$ then

$$V_n = \frac{1}{p^{n-1} - 1}\left[G_n^p - \sum \alpha_{e_1, \dots, e_t} V_1^{e_1} \dots V_t^{e_t} \right].$$

Then $V_n \equiv a_n^p$ modulo p and $H_*(BSU; Z_p) \cong$ Image $f_* = Z_p[G_2, \dots, G_n, \dots]$.

PROOF. Observe that for $n \not\equiv 1 \bmod p$, $G_{p,n} = Y_{p,n}$ modulo p^2. For $m > 1$, $G_{p,pm+1} \equiv Y_{p,n} - pa_1 a_m^p$ modulo p^2. Thus $G_n \equiv G'_n$ modulo p and $V_n \equiv V'_n$ modulo p for all $n \geqslant 2$. Since $H_*(BSU; Z_p) = H_*(BU; Z_{(p)}) \otimes Z_p$ it follows that $f_*(H_*(BSU; Z_p)) = f_*(H_*(BSU; Z_{(p)})) \otimes Z_p$. Thus our theorem follows from Theorem 4.2. Q.E.D.

THEOREM 4.4. $2a_{2n} + (-1)^n a_n^2 + \Sigma_{k=1}^{n-1}(-1)^k 2a_k a_{2n-k}$ *is a polynomial generator for $H_*(BSU; R)$ if* 2 *is a unit in R.*

PROOF. The canonical map g: $BSp \to BU$ factors through BSU. Thus Image $g_* \subset$ Image f_*. By [2, 17–06], $2a_{2n} + (-1)^n a_n^2 + \Sigma_{k=1}^{n-1}(-1)^k 2a_k a_{2n-k}$ is in Image g_*. Q.E.D.

Observe that in Theorem 4.4 we can take R to be $Z_{(p)}$ or Z_p for a p an odd prime.

5. Polynomial generators for $H_*(BSO; Z_2)$. Recall [2, 17–13] that $H^*(BO; Z_2) = Z_2[w_1, \dots, w_n, \dots]$ where $w_n \in H^n(BO; Z_2)$ is the nth Steifel-Whitney class with coproduct $\Delta(w_n) = \Sigma_{k=0}^n w_k \otimes w_{n-k}$. Let $b_n = (w_1^n)^*$ in the dual basis of the basis of $H^*(BO; Z_2)$ of monomials in the Stiefel-Whitney classes. Then $H_*(BO; Z_2) = Z_2[b_1, \ldots, b_n, \ldots]$ with coproduct $\Delta(b_n) = \Sigma_{k=0}^n b_k \otimes b_{n-k}$. The canonical map $g\colon BSO \to BO$ induces the quotient map $Z_2[w_1, \ldots, w_n, \ldots] \to Z_2[w_1, \ldots, w_n, \ldots]/(w_1)$ in Z_2-cohomology. Thus we have the same algebraic situation as for BSU with Z_2-coefficients. The only difference is that the degrees are halved. Thus g_* is a monomorphism and $H_*(BSO; Z_2)$ is a polynomial algebra with one generator in each degree greater than one. From §4 we know how to pick polynomial generators for $H_*(BSO; Z_2)$ as specific polynomials in $b_1, \ldots, b_n, \ldots$.

THEOREM 5.1. *In $Z[b_1, \ldots, b_n, \ldots]$ we define elements G_n and then V_n by induction on $n \geqslant 2$ from the following formulas*:

$$G_{2^s} = 2b_{2^s} + \sum_{i=1}^{2^{s-1}-1} 2b_i b_{2^s-i} + b_{2^{s-1}}^2,$$

$$G_{2n+1} = b_{2n+1} + \sum_{i=1}^{n} b_i b_{2n-i+1} + b_1 b_n^2 \quad \textit{for } n > 1,$$

$$G_{2n} = b_{2n} + \sum_{i=1}^{n-1} b_i b_{2n-i} + \frac{1}{2}\left(V_n + b_n^2\right) \quad \textit{for } n > 1,$$

$$V_{2^s} = G_{2^s+1}.$$

If $n \neq 2^s$, $n \geqslant 3$ and $G_n = b_n + \Sigma \alpha_{e_1, \ldots, e_t} b_1^{e_1} \ldots b_t^{e_t}$ then

$$V_n = G_n^2 + \sum \alpha_{e_1, \ldots, e_t} V_1^{e_1} \ldots V_t^{e_t}.$$

Then $V_n \equiv b_n^2$ modulo 2 and $H_*(BSO; Z_2) \simeq$ Image $g_* = Z_2[G_2, \ldots, G_n, \ldots]$. Q.E.D.

The above formula for G_{2n+1} was observed by B. Gray [**3**].

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