

SHORTER NOTES

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AN INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. Our purpose is to obtain in an elementary way a sharp estimate on the derivative of a trigonometric polynomial of degree $< n$ at a point θ when the trigonometric polynomial has a known bound at the Chebyshev points and at θ .

PROPOSITION. *If T is a trigonometric polynomial of degree $< n$ and if $\cos n\theta \neq 0$, then*

$$(1) \quad |T'(\theta)| \leq \frac{n}{|\cos n\theta|} \left[|T(\theta)| + \max_{1 \leq k \leq 2n} \left| T\left(\frac{(2k-1)\pi}{2n}\right) \right| \right].$$

Moreover, equality holds in (1) when $T(\phi) = \sin n\phi - \tan n\theta \cos n\phi$.

Inequality (1) implies a well-known extension of Bernstein's theorem for trigonometric polynomials. (See [1, p. 211] or [4, p. 102].) Indeed, applying (1) with $\theta = 0$ and with $T(\phi)$ replaced by $[T(\theta + \phi) - T(\theta - \phi)]/2$ and observing that $-(2k-1)\pi/(2n)$ is 2π less than $(2l-1)\pi/(2n)$, where $l = 2n - k + 1$, we obtain

$$|T'(\theta)| \leq n \max_{1 \leq k \leq 2n} \left| T\left(\theta + \frac{(2k-1)\pi}{2n}\right) \right|.$$

PROOF. Given θ , let M be the expression in brackets in (1) and put $S(\phi) = T(\theta + \phi) - T(\theta)$. Then S is a trigonometric polynomial of degree $< n$ such that $S(0) = 0$ and $|S(2\theta_k)| \leq M$ for $1 \leq k \leq 2n$, where $2\theta_k = (2k-1)\pi/(2n) - \theta$. It suffices to show that

$$(2) \quad |S'(0)| \leq nM/|\cos n\theta|.$$

Put $m = 2n$. We first observe that by the Lagrange interpolation formula, if $p(x) = a_0 + a_1x + \cdots + a_{m-1}x^{m-1}$ then

$$(3) \quad a_{m-1} = \sum_{k=1}^m \frac{p(x_k)}{\prod_{j \neq k} (x_k - x_j)},$$

Received by the editors March 1, 1981.

1980 *Mathematics Subject Classification.* Primary 26D05, 26D10.

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 0002-9939/82/0000-0034/\$01.50

where $x_k = \cot \theta_k$ for $1 \leq k \leq m$. In particular, taking $p(x) = \operatorname{Im}(x + i)^m$ in (3) and observing that

$$p(x_k) = \operatorname{Im}(e^{i\theta_k}/\sin \theta_k)^m = (-1)^{k-1} \csc^m \theta_k \cos n\theta,$$

we obtain

$$(4) \quad \frac{m}{|\cos n\theta|} = \sum_{k=1}^m \frac{\csc^m \theta_k}{\prod_{j \neq k} |x_k - x_j|}$$

since the sign of $\prod_{j \neq k} (x_k - x_j)$ alternates as k increases. Now by [3, p. 337], we may write $S(2\phi) = (\cos^m \phi)q(\tan \phi)$, where q is a polynomial of degree $< m$. Clearly $q(0) = 0$ and $2S'(0) = q'(0)$. Letting $p(x) = x^m q(\frac{1}{x})$, we see that $|p(x_k)| \leq M \csc^m \theta_k$ for $1 \leq k \leq m$ and that (3) holds with $a_{m-1} = q'(0)$. Thus (2) follows from (4). (Compare [5].)

A related result is obtained in [2, Theorem 2].

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