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SHORTER NOTES

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AN INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

LAWRENCE A. HARRIS

ABSTRACT. Our purpose is to obtain in an elementary way a sharp estimate on the derivative of a trigonometric polynomial of degree $\leqslant n$ at a point θ when the trigonometric polynomial has a known bound at the Chebyshev points and at θ.

PROPOSITION. *If T is a trigonometric polynomial of degree $\leqslant n$ and if* $\cos n\theta \neq 0$, *then*

$$|T'(\theta)| \leqslant \frac{n}{|\cos n\theta|}\left[|T(\theta)| + \max_{1 \leqslant k \leqslant 2n}\left|T\left(\frac{(2k-1)\pi}{2n}\right)\right|\right]. \tag{1}$$

Moreover, equality holds in (1) *when* $T(\phi) = \sin n\phi - \tan n\theta \cos n\phi$.

Inequality (1) implies a well-known extension of Bernstein's theorem for trigonometric polynomials. (See [**1**, p. 211] or [**4**, p. 102].) Indeed, applying (1) with $\theta = 0$ and with $T(\phi)$ replaced by $[T(\theta + \phi) - T(\theta - \phi)]/2$ and observing that $-(2k-1)\pi/(2n)$ is 2π less than $(2l-1)\pi/(2n)$, where $l = 2n - k + 1$, we obtain

$$|T'(\theta)| \leqslant n \max_{1 \leqslant k \leqslant 2n}\left|T\left(\theta + \frac{(2k-1)\pi}{2n}\right)\right|.$$

PROOF. Given θ, let M be the expression in brackets in (1) and put $S(\phi) = T(\theta + \phi) - T(\theta)$. Then S is a trigonometric polynomial of degree $\leqslant n$ such that $S(0) = 0$ and $|S(2\theta_k)| \leqslant M$ for $1 \leqslant k \leqslant 2n$, where $2\theta_k = (2k-1)\pi/(2n) - \theta$. It suffices to show that

$$|S'(0)| \leqslant nM/|\cos n\theta|. \tag{2}$$

Put $m = 2n$. We first observe that by the Lagrange interpolation formula, if $p(x) = a_0 + a_1x + \cdots + a_{m-1}x^{m-1}$ then

$$a_{m-1} = \sum_{k=1}^{m} \frac{p(x_k)}{\prod_{j \neq k}(x_k - x_j)}, \tag{3}$$

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where $x_k = \cot \theta_k$ for $1 \leqslant k \leqslant m$. In particular, taking $p(x) = \operatorname{Im}(x + i)^m$ in (3) and observing that

$$p(x_k) = \operatorname{Im}\left(e^{i\theta_k}/\sin \theta_k\right)^m = (-1)^{k-1} \csc^m \theta_k \cos n\theta,$$

we obtain

$$\frac{m}{|\cos n\theta|} = \sum_{k=1}^{m} \frac{\csc^m \theta_k}{\prod_{j \neq k} |x_k - x_j|} \tag{4}$$

since the sign of $\prod_{j \neq k}(x_k - x_j)$ alternates as k increases. Now by [**3**, p. 337], we may write $S(2\phi) = (\cos^m \phi) q(\tan \phi)$, where q is a polynomial of degree $\leqslant m$. Clearly $q(0) = 0$ and $2S'(0) = q'(0)$. Letting $p(x) = x^m q(\frac{1}{x})$, we see that $|p(x_k)| \leqslant M \csc^m \theta_k$ for $1 \leqslant k \leqslant m$ and that (3) holds with $a_{m-1} = q'(0)$. Thus (2) follows from (4). (Compare [**5**].)

A related result is obtained in [**2**, Theorem 2].

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506