UNIQUENESS AND QUASI-MEASURES ON
THE GROUP OF INTEGERS OF A P-SERIES FIELD

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Abstract. Let $G$ be the group of integers of a $p$-series field and suppose that $S$ is a
character series on $G$. If $N_1, N_2, \ldots$ is any sequence of integers and if $S_{N_j} \to 0$ a.e.
on $G$, as $j \to \infty$, then $S$ will be the zero series provided $S$ never diverges
unboundedly.

Let $G$ denote the group of integers of a $p$-series field, where $p$ is a prime $> 2$.
Thus, any element $\bar{x} \in G$ can be represented as a sequence $\{x_i\}_{i=0}^\infty$ with $0 < x_i < p$
for each $i > 0$. Moreover, the dual group $\{\psi_m\}_{m=0}^\infty$ of $G$ can be described by the
following process. If $m$ is a nonnegative integer with $m = \sum_{k=0}^\infty \alpha_k p^k$, $0 < \alpha_k < p$
for each $k$, and if $\bar{x} \in G$ then

$$
\psi_m(\bar{x}) = \prod_{k=0}^\infty \phi_k^m(\bar{x}),
$$

where for each integer $k > 0$ and for each $\bar{x} = \{x_i\} \in G$, the function $\phi_k$ is defined
by

$$
\phi_k(\bar{x}) = \exp(2\pi i x_k / p).
$$

In the case that $p = 2$, the group $G$ is the dyadic group introduced by Fine [2] and
the functions $\{\psi_m\}_{m=0}^\infty$ are the Walsh-Paley functions. A detailed account of these
groups and basic properties can be found in [5].

Denote the partial sums of a character series $S = \sum_{m=0}^\infty a_m \psi_m$ by

$$
S_N = \sum_{m=0}^{N-1} a_m \psi_m, \quad N = 1, 2, \ldots .
$$

Vilenkin [6] has shown that if $S_N \to 0$ everywhere on $G$ as $N \to \infty$, then $S$ is the
zero series, i.e., $a_m = 0$ for $m = 0, 1, \ldots$. When $N$ is replaced by $p^N$ and conver-
gence is relaxed on a countable subset of $G$, a growth condition is usually necessary
to retain uniqueness. For example, in [8] we saw that if $S_N$ converges to an integrable $f$ on all but countably many points in $G$, and if $p^{-N}S_N \to 0$ everywhere
on $G$ as $N \to \infty$, then $S$ is the $G$-Fourier series of $f$. The second hypothesis of this
result cannot be relaxed at a single point $\bar{x}_0 \in G$ where $|S_{p^N}(\bar{x}_0)| \to +\infty$ as

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Indeed, if $D = \sum_{m=0}^{\infty} \psi_m$ represents the Dirichlet kernel on $G$, then since

$$D_{p^N}(x) = \begin{cases} p^N & \text{if } \sum_{i=0}^{k-1} x p^{-i-1} < p^{-N}, \\ 0 & \text{otherwise}, \end{cases}$$

it is clear that both $D_{p^N}(x)$ and $p^{-N}D_{p^N}(x)$ converge to zero for $x \neq 0$ as $N \to \infty$ but $D$ is not the zero series. It is not yet known how far the first hypothesis can be relaxed (see [7]). One would expect that “convergence off a countable set” could be replaced with “convergence a.e.” but even in the case $p = 2$ this has not been done.

If one strengthens the second hypothesis to condition (4) below, then convergence a.e. can be used. Indeed, we shall prove the following result.

**Theorem.** Suppose that $S = \sum_{m=0}^{\infty} a_m \psi_m$ and that $(m_j)_{j=1}^{\infty}$ is a subsequence of the natural numbers. If $S_{p^N} \to 0$ a.e. on $G$ as $j \to \infty$, and if

$$\limsup_{j \to \infty} |S_{p^N}(\bar{x})| < \infty, \quad \bar{x} \in G,$$

then $S$ is the zero series.

Techniques used to establish uniqueness for Walsh series fall into two categories: proof by a Haar series argument (e.g., [1]), and proof by differentiation (e.g., [2] and [4]). Neither of these techniques seem suited to prove the theorem above.

Our technique introduces a fresh viewpoint, and uses quasi-measures (defined below) as a crutch for carrying out the necessary calculations. Recall that the topology of $G$ has a base at 0 which consists of closed/open subgroups $G_n$ whose Haar measure $m(G_n)$ equals $p^{-n}$, $n > 0$. We shall denote $G_0 \equiv G$ by $I(0,0)$, and for each integer $n > 0$ we shall denote the cosets of $G_n$ by $I(k,n)$, $0 < k < p^n$. The collection of sets $I(k,n)$, $0 < k < p^n$, $n = 0, 1, \ldots$, will be denoted by $\mathcal{I}$. Observe once and for all that $I(k_1,n) \cap I(k_2,n) = \emptyset$ for $k_1 \neq k_2$, that $m(I(k,n)) = p^{-n}$, that (reordering if necessary)

$$I(k,n) = \bigcup_{l=0}^{kp} I(l,n+1)$$

and that each $\psi_l$ is constant on each $I(k,n)$ when $l < p^n$. A set function $\mu$ defined on $\mathcal{I}$ is said to be a quasi-measure if

$$\mu(I(k,n)) = \sum_{l=0}^{kp} \mu(I(l,n+1)).$$

Clearly, every Borel measure on $G$ is also a quasi-measure.

Fix integers $k$ and $n$ with $0 < k < p^n$. By an argument similar to that found in [3], one can show that if $\lambda$ is a Borel measure on $G$, and if $S$ is its Fourier-Stieltjes series, then

$$\lambda(I(k,n)) = \lim_{N \to \infty} \int_{I(k,n)} S_N \ dm.$$

Since $\int_{I(k,n)} \psi_l \ dm = 0$ for $l > p^n$ and since $S_{p^N}$ is constant on $I(k,n)$, it follows that $\lambda(I(k,n)) = p^{-n}S_{p^N} \psi_l(x)$ for any choice of $x \in I(k,n)$. Thus we are led to associate
with any character series $S$ the set function $\mu$ defined on $\mathcal{G}$ by
\begin{equation}
\mu(I(k, n)) \equiv p^{-n}\mathcal{S}_{\nu}(\overline{x}), \quad \overline{x} \in I(k, n).
\end{equation}

A routine calculation (recall that the sum of $p$th roots of unity is zero) establishes that this set function $\mu$ is a quasi-measure. Moreover, since the characters of $G$ are orthogonal it is clear that a necessary and sufficient condition for a character series $S$ to be zero is that its associated quasi-measure satisfies $\mu(I) = 0$ for all $I \in \mathcal{G}$.

We are now prepared to prove the theorem. Indeed, if $S$ is the given series and if $\mu$ is its associated quasi-measure, we need only show that $\mu \equiv 0$. We shall actually show that $\mu(G) = 0$. The same argument can be used to show that $\mu(I) = 0$ for all $I \in \mathcal{G}$ and thus complete the proof.

We assume for simplicity that $m_j = j$. Fix $0 < \epsilon < 1$. By Egoroff's Theorem we can choose a subset $E_1$ of $G$ such that $m(E_1) > 1 - \epsilon$ and such that $\mathcal{S}_{\nu}$ converges uniformly to zero on $E_1$, as $N \to \infty$. Thus, given $\epsilon_1 > 0$ there exists an integer $N_1$ such that $|\mathcal{S}_{\nu}(\overline{x})| < \epsilon_1$ for $\overline{x} \in E_1$. But $\mathcal{S}_{\nu}$ is actually constant on sets of the form $I(k, N_1)$ so there exists a subset $Z_1$ of the positive integers such that $k \in Z_1$ implies $|\mathcal{S}_{\nu}(\overline{x})| < \epsilon_1$ for $\overline{x} \in I(k, N_1)$.

Observe that if $|Z_1|$ represents the cardinality of $Z_1$ then
\begin{equation}
|Z_1| \cdot p^{-N_1} = \sum_{k \in Z_1} m(I(k, N_1)) > m(E_1) > 1 - \epsilon.
\end{equation}

Thus $1 - |Z_1| \cdot p^{-N_1} < \epsilon$. Now, let $k_1$ be an integer which satisfies
\begin{equation}
|\mu(I(k_1, N_1))| = \max_{k \notin Z_1} |\mu(I(k, N_1))|
\end{equation}

and observe that $\mathcal{S}_{\nu} \to 0$ a.e. on $I(k_1, N_1)$ as $N \to \infty$. Thus, given $\epsilon_2 > 0$ we can choose a subset $E_2$ of $I(k_1, N_1)$ such that $m(E_2) > (1 - \epsilon)p^{-N_1}$ and choose an integer $N_2$ such that $|\mathcal{S}_{\nu}(\overline{x})| < \epsilon_2$ for $\overline{x} \in E_2$.

Continuing in this manner, given any positive integer $j$ and $\epsilon_j > 0$ we can choose positive integers $N_j$, $k_j$, subsets $Z_j$ of natural numbers and $E_j$ of $G$ such that
\begin{equation}
1 - |Z_j| \cdot p^{-N_j} < \epsilon,
\end{equation}

if $k \in Z_j$ then $|\mathcal{S}_{\nu}(\overline{x})| < \epsilon_j$ for $\overline{x} \in I(k, N_1 + \cdots + N_j)$

and
\begin{equation}
|\mu(I(k_j, N_1 + \cdots + N_j))| = \max_{k \notin Z_j} |\mu(I(k, N_1 + \cdots + N_j))|: k \notin Z_j
\end{equation}

and $I(k, N_1 + \cdots + N_j) \subseteq I(k_{j-1}, N_1 + \cdots + N_{j-1})$.

To estimate $\mu(G)$, observe by (6) that $|\mu(G)| < T_1 + \tilde{T}_1$ where
\begin{equation}
T_1 = \sum \{|\mu(I(k, N_1))|: k \in Z_1\}
\end{equation}

and
\begin{equation}
\tilde{T}_1 = \sum \{|\mu(I(k, N_1))|: 0 < k < p^{N_1}, k \notin Z_1\}.
\end{equation}

To estimate $T_1$ we observe that $|Z_1| \cdot p^{-N_1} < 1$ follows from the fact that there are at most $p^{N_1}$ sets of the form $I(k, N_1)$. Therefore, by (7) and (9) we have that $T_1 < \epsilon_1$. 

To estimate $\tilde{T}_1$ we apply (10) to conclude that

$$\tilde{T}_1 \leq (p^{-N_1} - |Z_1|)\mu(I(k_1, N_1)).$$

Thus

$$|\mu(G)| \leq \epsilon_1 + (p^{-N_1} - |Z_1|)\mu(I(k_1, N_1)).$$

But (6), (7), (9), and (10) can be used to estimate $|\mu(I(k_1, N_1))|$ by summing over those intervals $I(k, N_1 + N_2)$ which are contained in $I(k_1, N_1)$ and separating the indices $k \in \mathbb{Z}_2$ from $k \notin \mathbb{Z}_2$. Again $|Z_2|p^{-N_2} < 1$ follows from the fact that there are at most $p^{N_2}$ sets of the form $I(k, N_1 + N_2)$ contained in $I(k_1, N_1)$.

If we continue breaking up $|\mu(I(k_j, N_1 + \cdots + N_j))|$ in this manner we arrive at the following inequality:

$$|\mu(G)| \leq \epsilon_1 + \epsilon_2(p^{-N_1} - |Z_1|) + \cdots + \epsilon_j \prod_{l=1}^{j-1} (p^{N_l} - |Z_l|) + |\mu(I(k_j, N_1 + \cdots + N_j))| \cdot \prod_{j=1}^{j} (p^{N_j} - |Z_j|),$$

$j = 1, 2, \ldots$. Set $\epsilon_1 = \epsilon/2$ and for each integer $j > 1$ set

$$\epsilon_j = 2^{-j}\epsilon \left( \prod_{l=1}^{j-1} (p^{N_l} - |Z_l|) \right)^{-1}. $$

It follows from (8) and (11) that

$$|\mu(G)| \leq \epsilon(1 - 2^{-j}) + \epsilon(p^{N_1} + \cdots + N_j) \mu(I(k_j, N_1 + \cdots + N_j)).$$

Observe by construction that the collection of compact sets

$$\{I(k_j, N_1 + \cdots + N_j)\}_{j=1}^{\infty}$$

is nested. Consequently, we can choose a point $\bar{x}_0$ which belongs to $I(k_j, N_1 + \cdots + N_j)$ for all integers $j \geq 1$. By (7), then, the estimate (12) becomes

$$|\mu(G)| \leq \epsilon + \epsilon|S_{p,n}(\bar{x}_0)|.$$

Our simplifying assumption turns (4) into $\limsup_{N \to \infty} S_{p,n}(\bar{x}_0) = A < \infty$. Thus

$$|\mu(G)| \leq \epsilon + A\epsilon.$$

In particular, if we let $\epsilon \to 0$ we obtain $\mu(G) = 0$, thus completing the proof of the theorem.

**Bibliography**


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