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PSEUDOHOLOMORPHIC FUNCTIONS WITH NONANTIHOLOMORPHIC CHARACTERISTICS

AKIRA KOOHARA

ABSTRACT. Let $\kappa(z) \in C^\infty(\Omega)$ and $\|\kappa\| < 1$. Necessary and sufficient conditions for the system of equations $\partial\bar{f} = \kappa(z)\partial f$ to be locally plentiful are given, and under them a representation of κ also is given.

1. Introduction. Let Ω be a domain in C^n and let $C^\infty(\Omega)$ denote the space of infinitely differentiable complex valued functions on Ω. Let a and b be in $C^\infty_{(1,0)}(\Omega)$, the space of C^∞ differential forms of type (1, 0) on Ω. Now, consider the R-linear mapping ν: $C^\infty(\Omega) \to C^\infty_{(1,0)}(\Omega)$ defined by $\nu(f) = \partial\bar{f} - \bar{f}a - fb$ for $f \in C^\infty(\Omega)$, where ∂ denotes the operator $\sum_{j=1}^n \partial_j dz_j$ with $\partial_j = \partial/\partial z_j$. Then, Ker ν, the kernel of the map ν, is an R-submodule of $C^\infty(\Omega)$.

Quite recently there has been increasing interest in Ker ν, whose elements are called generalized analytic functions of several complex variables (see [**5, 6, 7, 8**] and references cited in [**7**]). We call the equation $\nu(f) = 0$ the generalized Cauchy-Riemann equation.

Magomedov and Paramodov [**6**] introduced the idea of the plentifulness of Ker ν to obtain the integrability conditions of the equation $\nu(f) = 0$ with $a = 0$ on Ω. When $\dim_R$ Ker ν is infinite on Ω, Ker ν is said to be plentiful on Ω. The plentifulness on Ω leads to a complex foliation of codimension one of Ω determined by the form b. The null sets of generalized analytic functions are leaves of this foliation.

In [**3**] the author treated generalized analytic functions under the conditions on b such that a complex foliation of codimension one of Ω follows from them.

In this paper we are concerned with the R-linear mapping α: $C^\infty(\Omega) \to C^\infty_{(1,0)}(\Omega)$ defined by

$$\alpha(f) = \sum_{j=1}^n \left\{ \kappa(z)\partial_j f - \partial_j \bar{f} \right\} dz_j, \quad \text{for } f, \kappa \in C^\infty(\Omega).$$

Ker α also is an R-submodule of $C^\infty(\Omega)$ as the map ν. The equation $\alpha(f) = 0$ was investigated by S. Hitotumatu [**2**] and by the author [**4**]. The former used function-theoretic methods and the latter differential equation-theoretical ones.

In [**4**], given some conditions upon the coefficient κ, we discussed properties of elements of Ker α (which we call pseudoholomorphic functions with characteristic

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κ) similar to those of holomorphic functions and obtained a local representation theorem of such functions.

Now, by using the results in [6, 7] we can obtain necessary and sufficient conditions for Ker α to be plentiful, because the system of elliptic differential equations $\alpha(f) = 0$ can be reduced to the system of type $\nu(g) = 0$ by $f = g + \bar{\kappa}\bar{g}$. However, the methods and results of [6, 7] are not effectual ones to clarify completely structures of Ker α.

The purpose of this paper is to investigate relations between $\dim_R$ Ker α and coefficient κ, and to give a local representation of κ in the case where Ker α is plentiful.

2. Preliminaries and notations. Since Ker α is an R-submodule of $C^\infty(\Omega)$, following Magomedov and Paramodov, when $\dim_R$ Ker α is infinite, we say that Ker α or the system of differential equations

$$\partial_j \bar{f} = \kappa(z)\partial_j f, \qquad j = 1, 2, \ldots, n, \tag{2.1}$$

is plentiful on Ω.

To attain our objective we need a few assumptions on characteristic κ. First we assume $\|\kappa\| = \sup_\Omega |\kappa(z)| < 1$.

If $\partial\kappa$ vanishes on an open subset U in Ω, then, considering the restriction of α to U denoted by $\alpha|_U$, we can see that $\mathrm{Ker}(\alpha|_U)$ is plentiful [4]. Or, if κ vanishes on U, then (2.1) is the Cauchy-Riemann equations on U. By these reasons we may assume that, for nowhere dense subsets E_1 and E_2 of Ω,

$$\kappa \neq 0 \quad \text{on } \Omega \setminus E_1, \qquad \partial\kappa \neq 0 \quad \text{on } \Omega \setminus E_2. \tag{2.2}$$

Let $C^\infty_{(p,q)}(\Omega)$ denote the space of C^∞ differential forms of type (p, q) on Ω.

We shall define the R-linear mapping α^* of $C^\infty(\Omega)$ into $C^\infty_{(0,1)}(\Omega)$ by

$$\alpha^*(f) = \sum_{j=1}^{n} \left\{ \overline{\kappa(z)}\, \overline{\partial_j f} - \bar{\partial} f \right\} d\bar{z}_j = \overline{\alpha(f)}\,.$$

Then, we may regard α and α^* as R-linear differential operators of first order on $C^\infty(\Omega)$.

Let σ be a vector field on U and f in $C^\infty(U)$. When $\sigma f = 0$ and $\sigma\bar{f} = 0$ on U, we say that the vector field σ is tangential to f. And when, for every $f \in \mathrm{Ker}(\alpha|_U)$, σ is tangential to f, we say that σ is tangential to $\mathrm{Ker}(\alpha|_U)$.

To seek vector fields tangential to $\mathrm{Ker}(\alpha|_U)$, we need to construct three C-linear mappings β, $\bar{\beta}$ and θ: $C^\infty(\Omega) \to C^\infty_{(p,q)}(\Omega)$ such that their kernels contain Ker α and Ker α^*.

Rewriting the map α by using ∂, we have $\alpha(f) = \kappa(z)\partial f - \partial\bar{f}$ for $f \in C^\infty(\Omega)$.

Then we have readily the C-linear mapping $\partial\alpha$: $C^\infty(\Omega) \to C^\infty_{(2,0)}(\Omega)$ defined by

$$\partial\alpha(f) = \partial\kappa \wedge \partial f \quad \text{for } f \in C^\infty(\Omega). \tag{2.3}$$

We put

$$\beta = \partial\alpha \quad (= \partial\kappa \wedge \partial). \tag{2.4}$$

Then we obtain

$$\beta(\bar{f}) = \kappa\partial\alpha(f) - \partial\kappa \wedge \alpha(f). \tag{2.5}$$

We thus define the mapping $\bar{\beta}$: $C^\infty(\Omega) \to C^\infty_{(0,2)}(\Omega)$ as

$$\bar{\beta}(f) = \overline{\beta(\bar{f})} \quad \text{for } f \in C^\infty(\Omega). \tag{2.6}$$

From the definition of α^* and (2.3)–(2.6) we obtain

$$\operatorname{Ker} \alpha^* = \operatorname{Ker} \alpha, \qquad \operatorname{Ker} \alpha \subset \operatorname{Ker} \beta = \operatorname{Ker} \bar{\beta}. \tag{2.7}$$

Lastly, we want to construct a mapping θ of $C^\infty(\Omega)$ into $C^\infty_{(2,1)}(\Omega)$. To do this, we need the identity: for $f \in C^\infty(\Omega)$

$$\begin{aligned}\bar{\kappa}\{\bar{\partial}\partial\alpha(f) + \partial\kappa \wedge \partial\alpha^*(f) - \bar{\kappa}\bar{\partial}\beta(\bar{f}) - \bar{\kappa}\bar{\partial}\partial\kappa \wedge \alpha(f)\} - \partial\kappa \wedge \partial\bar{\kappa} \wedge \alpha^*(f)\\ = \bar{\kappa}(1 - |\kappa|^2)\bar{\partial}\partial\kappa \wedge \partial f + \partial\kappa \wedge \partial\bar{\kappa} \wedge \bar{\partial} f.\end{aligned}$$

Then, θ is defined as follows:

$$\theta(f) = \bar{\kappa}(1 - |\kappa|^2)\bar{\partial}\partial\kappa \wedge \partial f + \partial\kappa \wedge \partial\bar{\kappa} \wedge \bar{\partial} f \quad \text{for } f \in C^\infty(\Omega), \tag{2.8}$$

where $\bar{\partial} = \sum_{j=1}^n \bar{\partial}_j d\bar{z}_j$, $\bar{\partial}_j = \partial/\partial\bar{z}_j$.

The three mappings defined above may be regarded as C-linear differential operators of first order on $C^\infty(\Omega)$.

It follows from (2.7) and the above identity that

$$\operatorname{Ker} \alpha \subset \operatorname{Ker} \theta. \tag{2.9}$$

For the purposes of later convenience, we now express (2.4) and (2.8) in terms of coordinates in C^n.

We put

$$\begin{aligned}\kappa_i &= \partial_i\kappa, \qquad \beta_{ij} = \kappa_i\partial_j - \kappa_j\partial_i,\\ \gamma_{ijk} &= (\bar{\partial}_k\kappa_i)\partial_j - (\bar{\partial}_k\kappa_j)\partial_i,\\ \theta_{ijk} &= \bar{\kappa}(1 - |\kappa|^2)\gamma_{ijk} + \beta_{ij}(\bar{\kappa})\bar{\partial}_k.\end{aligned}$$

From now on, the indices i, j and k (with or without subscripts) run over the set $\{1, 2, \ldots, n\}$ unless specifically stated otherwise.

Then we have

$$\begin{aligned}\beta &= \partial\kappa \wedge \partial = \sum_{i<j} \{(\partial_i\kappa)\partial_j - (\partial_j\kappa)\partial_i\}\, dz_i \wedge dz_j,\\ \theta &= \sum_{i<j,\,k} \theta_{ijk} dz_i \wedge dz_j \wedge d\bar{z}_k.\end{aligned}$$

In the following section we shall prove that on Ω

$$\bar{\partial}\partial\kappa \wedge \partial\kappa = 0, \qquad \partial\kappa \wedge \partial\bar{\kappa} = 0. \tag{2.10}$$

In terms of coordinates of C^n we rewrite the left sides of (2.10).

$$\begin{aligned}\bar{\partial}\partial\kappa \wedge \partial\kappa &= \sum_{i<j,\,k} \gamma_{ijk}(\kappa)\, dz_i \wedge dz_j \wedge d\bar{z}_k,\\ \partial\kappa \wedge \partial\bar{\kappa} &= \sum_{i<j} \beta_{ij}(\bar{\kappa})\, dz_i \wedge dz_j.\end{aligned}$$

3. Necessary conditions for plentifulness. Let w be a nonconstant pseudoholomorphic function on Ω. By the unique continuation property for pseudoholomorphic functions [4], we have a nowhere dense subset E_3 of Ω such that $\partial w \neq 0$ on $\Omega \setminus E_3$. If we put $E = E_1 \cup E_2 \cup E_3$, $\kappa \neq 0$, $\partial\kappa \neq 0$ and $\partial w \neq 0$ on $\Omega \setminus E$. Since $\partial\kappa \wedge \partial w = 0$ on Ω, for any $z \in \Omega \setminus E$ there is a number i' such that $\partial_{i'}\kappa \neq 0$ and $\partial_{i'}w \neq 0$. Though we must prove (2.10) about each point of $\Omega \setminus E$, it is enough to prove it about a specific point. We may assume without loss of generality that if $0 \in \Omega \setminus E$, then $w(0) = 0$, $\partial_n\kappa(0) \neq 0$ and $\partial_n w(0) \neq 0$.

To prove (2.10) we use the following special change of variables on a small neighborhood U of the origin

$$\xi_j = z_j, \qquad j = 1, 2, \ldots, n-1, \quad \zeta = w(z). \tag{3.1}$$

This is nonsingular because w satisfies (2.1).

We put $\partial'_j = \partial/\partial\xi_j$, $\bar\partial'_j = \partial/\partial\bar\xi_j$, $\Delta = \bar\kappa(1 - |\kappa|^2)$ and $\kappa_{i\bar k} = \bar\partial_k\kappa_i$. Moreover we denote by $[\ ]''$ the functions into which ones in $[\ ]$ are transformed by (3.1). If we note (2.9), i.e. $\theta_{ijk}(f) = 0$ for any $f \in \operatorname{Ker}\alpha$, θ_{ijk} are transformed into the following on U:

$$\begin{aligned}
[\theta_{ijk}]'' &= \Delta''\{[\kappa_{i\bar k}]''\partial'_j - [\kappa_{j\bar k}]''\partial'_i\} + [\beta_{ij}(\bar\kappa)]''\bar\partial'_k + [\theta_{ijk}(\bar w)]''\partial_{\bar\zeta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \neq n, j \neq n, k \neq n, \\
[\theta_{njk}]'' &= [\Delta\kappa_{n\bar k}]''\partial'_j + [\beta_{nj}(\bar\kappa)]''\bar\partial'_k + [\theta_{njk}(\bar w)]''\partial_{\bar\zeta} \quad \text{for } j \neq n, k \neq n, \\
[\theta_{ink}]'' &= -[\Delta\kappa_{n\bar k}]''\partial'_i + [\beta_{in}(\bar\kappa)]''\bar\partial'_k + [\theta_{ink}(\bar w)]''\partial_{\bar\zeta} \quad \text{for } i \neq n, k \neq n, \\
[\theta_{ijn}]'' &= \Delta''\{[\kappa_{i\bar n}]''\partial'_j - [\kappa_{j\bar n}]''\partial'_i\} + [\theta_{ijn}(\bar w)]''\partial_{\bar\zeta} \quad \text{for } i \neq n, j \neq n.
\end{aligned} \tag{3.2}$$

LEMMA 1. *If* Ker α *has an element W linearly independent of w, then the vector fields θ_{ijk} are tangential to w.*

PROOF. It is sufficient to show $\theta_{ijk}(\bar w) = 0$. Assume there are a point z' and numbers i', j', k' such that $\theta_{i'j'k'}(\bar w) \neq 0$ at z'. We may regard z' as the origin and, shrinking U mentioned above if necessary, assume that $\theta_{i'j'k'}(\bar w) \neq 0$ on U. Using the coordinates introduced in (3.1), we obtain, on the image of U by (3.1), $\partial'_j W'' = 0$, $\bar\partial'_j W'' = 0$ $(j = 1, \ldots, n-1)$ and $\partial_{\bar\zeta} W'' = 0$, where we use the relation derived from (3.2), $\theta_{i'j'k'}(W) = [\theta_{i'j'k'}(w)]''\partial_{\bar\zeta} W''$. Therefore we see W'' depends only on ζ and is holomorphic at 0.

However, since $\alpha(W) = 0$, $[\kappa\partial_n w]''(\partial_\zeta W'' - \partial_{\bar\zeta}\overline{W}'') = 0$, and hence $\partial_\zeta W'' = \partial_{\bar\zeta}\overline{W}''$. Thus we obtain $W = aw + b$ on U, where $a > 0$ is a constant and $b \in C$, which contradicts the assumption.

Let S be any subset of Ω. The set $N(S)$ of those vector fields on S which are tangential to w is a $C^\infty(S)$-submodule of the $C^\infty(S)$-module $M(S)$ consisting of all vector fields on S.

If $f \in \operatorname{Ker}\alpha$ is nonconstant, nonempty level sets $\{z \in \Omega | f(z) = \text{const.}\}$ are $(n-1)$-dimensional complex submanifolds except the set of nonordinary points of f [2].

By virtue of (2.7) every vector field β_{ij} is tangential to Ker α. If $0 \in \Omega \setminus E$, letting β_i denote β_{in} ($i = 1, \ldots, n-1$), we see from the above-mentioned that $\{\beta_i, \bar{\beta}_i\}$ span $N(U)$.

We say Ker α is trivial when it is C itself.

LEMMA 2. *Under the same assumption as in Lemma* 1, (2.10) *holds on* Ω.

PROOF. Lemma 1 shows $\theta_{ijk} \in N(\Omega)$. Assume $0 \in \Omega \setminus E$. Then θ_{ijk} can be written by linear combinations of β_s, $\bar{\beta}_s$ ($s = 1, \ldots, n-1$) with coefficients in $C^\infty(U)$. If a^s_{ijk}, b^s_{ijk} denote the coefficients of β_s, $\bar{\beta}_s$, we have:

$$\begin{aligned} &\text{For } 1 \leqslant i < j \leqslant n-1 \text{ and each } k, \\ &a^s_{ijk} = 0, \qquad 1 \leqslant s \leqslant n-1, s \neq i, j, \\ &\kappa_n a^i_{ijk} = \Delta\kappa_{j\bar{k}}, \quad \kappa_n a^j_{ijk} = -\Delta\kappa_{i\bar{k}}, \quad \sum_{s=1}^{n-1} a^s_{ijk}\kappa_s = 0. \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\text{For } 1 \leqslant i < j = n \text{ and each } k, \\ &a^s_{ink} = 0, \qquad 1 \leqslant s \leqslant n-1, s \neq i, \\ &\kappa_n a^i_{ink} = \Delta\kappa_{n\bar{k}}, \qquad \kappa_i a^i_{ink} = \Delta\kappa_{i\bar{k}}. \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\text{For } 1 \leqslant k \leqslant n-1 \text{ and each } i, j, \\ &b^{s'}_{ijk} = 0, \qquad 1 \leqslant s' \leqslant n-1, s' \neq k, \\ &\bar{\kappa}_k b^k_{ijk} = 0, \qquad \bar{\kappa}_n b^k_{ijk} = -\beta_{ij}(\bar{\kappa}). \end{aligned} \tag{3.5}$$

For $k = n$ and each i, j, $\beta_{ij}(\bar{\kappa}) = 0$ by $\beta^{s'}_{ijn} = 0$, $1 \leqslant s' \leqslant n-1$.

If $k \neq n$, from (3.5) we need to consider the following two cases.

Case 1. For all k, $1 \leqslant k \leqslant n-1$, $\kappa_k = 0$ on U.

Case 2. For some k', $1 \leqslant k' \leqslant n-1$ and some point $z' \in U$, $\kappa_{k'} \neq 0$ at z'.

We prove the first part of (2.10) only in Case 1. Let there be an open subset V of U and some i', j' ($i' < j'$) such that $\beta_{i'j'}(\bar{\kappa}) \neq 0$. Then, $\beta_i = \partial_i$, $i = 1, \ldots, n-1$, and so

$$\bar{\partial}_k = \left[\beta_{i'j'}(\bar{\kappa})\right]^{-1}\left\{\theta_{i'j'k} + (\Delta\kappa_{j'\bar{k}})\beta_{i'}\right\} \in N(V).$$

Hence $N(V) = M(V)$, which contradicts the nontrivial Ker α.

We next prove the second part of (2.10). From (3.3) and (3.4) we have $\gamma_{ijk} = \kappa_{i\bar{k}}\kappa_j - \kappa_{j\bar{k}}\kappa_i = 0$ on U, which completes the proof.

COROLLARY. *Let U be an open subset of Ω. If on U either* (1) $\partial\kappa \wedge \partial\bar{\kappa} \neq 0$, $\bar{\partial}\partial\kappa \wedge \partial\kappa = 0$ *or* (2) $\partial\kappa \wedge \partial\bar{\kappa} = 0$, $\bar{\partial}\partial\kappa \wedge \partial\kappa \neq 0$, *then* Ker α *is trivial.*

PROOF. Let $w \in$ Ker α be nonconstant.

Case (1). $\bar{\partial}\partial\kappa \wedge \partial\kappa = 0$ leads to $\theta(w) = \partial\kappa \wedge \partial\bar{\kappa} \wedge \bar{\partial}w$. By (2.7) and (2.9), $\bar{\partial}w = 0$ on U. w is constant.

Case (2). $\partial\kappa \wedge \partial\bar{\kappa} = 0$ leads to $\theta(w) = \Delta\bar{\partial}\partial\kappa \wedge \partial w$. Since $\partial\kappa \wedge \partial w = 0$, $\theta(w) = c\Delta\bar{\partial}\partial\kappa \wedge \partial\kappa$ for some function $c \in C^\infty(U)$, and hence $\bar{\partial}\partial\kappa \wedge \partial\kappa = 0$ on U, which contradicts the assumption.

THEOREM 1. *For the system* (2.1) *to be plentiful on* Ω, *it is necessary that the characteristic* κ *fulfill the condition* (2.10) *on* Ω.

4. Sufficient conditions for plentifulness. We show the local validity of the converse of Theorem 1. As is readily verified, the first half of (2.10) is sufficient for the (1, 0)-form $\sum \kappa_j dz_j$ to determine a complex foliation of codimension one of Ω. The converse of this is not always valid (see, e.g. Example (ii) below).

LEMMA 3. *For a function $\kappa \in C^\infty(\Omega)$ satisfying $\partial\kappa \neq 0$ and* (2.10) *on Ω, there is locally a holomorphic function h such that $dh \wedge \partial\kappa = 0$, $dh \wedge \partial\bar{\kappa} = 0$ and $dh \neq 0$.*

PROOF. If we put $\omega = \partial\kappa$, by the first half of (2.10) $\bar{\partial}\omega = \rho \wedge \omega$ for a form $\rho \in C^\infty_{(0,1)}(\Omega)$. From this $\bar{\partial}\rho \wedge \omega = 0$, $\omega \neq 0$ leads to $\bar{\partial}\rho = 0$ on Ω, so that for each point of Ω there are a neighborhood U of that point and a function $g \in C^\infty(U)$ such that $\bar{\partial}g = \rho$. Putting $\tau = \omega \exp(-g)$, we see $\bar{\partial}\tau = 0$, which shows τ is a holomorphic form. By using $\omega = \tau \exp g$, we have $\partial g \wedge \tau + d\tau = 0$, and hence

$$\tau \wedge d\tau = 0, \quad \tau \neq 0 \qquad \text{on } U. \tag{4.1}$$

Let $H(U)$ denote the algebra of all holomorphic functions on U. We define $\tau = \sum \tau_j dz_j$ and $D_{ij} = \tau_i \partial_j - \tau_j \partial_i$, $\tau_j \in H(U)$. Then, by (4.1) we have a function h holomorphic on a neighborhood $V \subset U$ such that $dh \neq 0$ and $\tau \wedge dh = 0$ (i.e. D_{ij} is tangential to h). Thus the proof is complete.

LEMMA 4 [4, THEOREM 20]. *Assume that κ satisfies* (2.10) *and $\partial\kappa \neq 0$ on Ω. Then* (2.1) *is locally reduced to the equation of one variable*

$$\partial_{\bar{t}} F = \overline{K(t)\partial_t F}, \qquad |K| < 1, \tag{4.2}$$

where $K(t)$ is defined and of class $C^\infty(h(V))$.

PROOF. We have a holomorphic function h satisfying the conditions of Lemma 3 on an open subset V of Ω. We may assume h is the coordinate function z_n. Since $\partial\kappa \wedge dz_n = 0$, $\partial\bar{\kappa} \wedge dz_n = 0$, κ and $\bar{\kappa}$ are holomorphic in the other coordinates when fixing z_n, and so is a function of z_n alone. Then $\kappa = K(z_n)$ and, for any $f \in \operatorname{Ker} \alpha$, $\partial f \wedge dz_n = \partial \bar{f} \wedge dz_n = 0$, so $f = F(z_n)$. Thus equation (4.2) is obtained.

We now take a disk $\delta \subset\subset h(V)$. Then we have

LEMMA 5. *Equation* (4.2) *is plentiful on δ.*

PROOF. Let Δ_i $(i = 1, 2)$ be disks concentric with δ such that $\delta \subset\subset \Delta_1 \subset\subset \Delta_2$. We take a function $K_1(t) \in C^\infty$ on C as follows: $K_1(t)$ equals one on δ and zero outside Δ_1. Besides, it fulfills $0 \leqslant K_1(t) \leqslant 1$. Putting $L(t) = K_1(t)K(t)$, we have the equation

$$\partial_{\bar{t}} G = \overline{L(t)}\ \overline{\partial_t G}. \tag{4.3}$$

Consider the Dirichlet problem of (4.3) with the boundary conditions $\operatorname{Re} G(t) = g(t)$ on $\partial\Delta_2$, and $G(t_0) = 0$, $t_0 \in \Delta_2$, where $g(t)$ is a given real valued continuous function on Δ_2. This problem is solvable [1]. The plentifulness on δ of (4.2) is derived from the fact that $\dim_R C(\partial\Delta_2)$ is infinite and from the unique continuation property for solutions of equation (4.3). Thus we have the following

THEOREM 2. *If the characteristic κ satisfies* (2.10) *on Ω, then* Ker α *is locally plentiful.*

Using Lemma 3 and the proof of Lemma 4, we have a local representation of κ.

THEOREM 3. *There are locally a holomorphic function h and $K(t) \in C^\infty(\operatorname{img} h)$, $t = h(z)$, such that $\kappa = K \circ h$, $dh \neq 0$ and $K \neq 0$ if and only if κ satisfies* (2.10) *and* $\partial\kappa \neq 0$ *on* Ω.

EXAMPLES. (i) Consider $\kappa = \phi(z) + \overline{\psi(z)}$, where ϕ and ψ are holomorphic and $d\phi \wedge d\psi = 0$ on Ω. Then $\partial\kappa \wedge \partial\bar{\kappa} \neq 0$, even though $\bar{\partial}\partial\kappa \wedge \partial\kappa = 0$, so Ker α is trivial by the corollary to Lemma 2.

(ii) We consider (2.1) on a small neighborhood U of the origin in C^2 such that $w(z)$ is well defined on U by the following equation:

$$(\bar{w} + z_1 + z_2)^2 - 2w = 2z_2, \qquad w(0) = 0. \tag{4.4}$$

Putting $\kappa = w(z) + \bar{z}_1 + \bar{z}_2$ (restricting U further if necessary) we can observe that Ker α is generated only by $w(z)$ (see also [**4**]). A simple computation shows that $\partial_1 w - \partial_2 w \neq 0$ on U, and so it is easy to obtain $\partial\kappa \wedge \partial\bar{\kappa} = (\partial_1 w - \partial_2 w)\, dz_1 \wedge dz_2 \neq 0$. Moreover we can also show

$$\bar{\partial}\partial\kappa \wedge \partial\kappa \neq 0 \quad \text{on } U. \tag{4.5}$$

(iii) We regard the function w defined by (4.4) as $\kappa(z)$. Evidently we have $\partial\kappa \wedge \partial\bar{\kappa} = 0$ and (4.5) on U. By the corollary to Lemma 2 Ker α is trivial.

(iv) Global plentifulness is not always true, even though κ satisfies (2.10) on Ω. The following example shows global plentifulness is valid.

If we take $\kappa = (2/3)(z_1^2 + z_2)$ on $\Omega = \{z \in C^2\colon |z_1^2 + z_2| < 1\}$, then we have easily, for any nonnegative integers m,

$$w = (z_1^2 + z_2)^{m+1}/(m+1) + (2/3)(\bar{z}_1^2 + \bar{z}_2)^{m+2}/(m+2).$$

It is trivial for Ker α to be plentiful on Ω.

In conclusion we are in a position to state relations between $\dim_R$ Ker α and κ. We define the following notations: $d = \dim_R$ Ker α, $\theta_1 = \partial\kappa \wedge \partial\bar{\kappa}$ and $\theta_2 = \bar{\partial}\partial\kappa \wedge \partial\kappa$.

(1) If $d \geqslant 3$, $\theta_1 = \theta_2 = 0$ on Ω.

(2) If $\theta_1 = \theta_2 = 0$ on Ω, $d = +\infty$ locally.

(3) If there is an open subset U of Ω on which either $\theta_1 \neq 0$, $\theta_2 = 0$ or $\theta_1 = 0$, $\theta_2 \neq 0$, then $d = 1$ (Ker α is trivial).

(4) If there is a point of Ω at which $\theta_1 \neq 0$, $\theta_2 \neq 0$, then $d \leqslant 2$.

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Department of Mathematics, Himeji Institute of Technology, Shosha 2167, Himeji, 671-22, Japan