PSEUDOHOLOMORPHIC FUNCTIONS WITH NONANTIHOLOMORPHIC CHARACTERISTICS

AKIRA KOOHARA

Abstract. Let \( k(z) \in C^\infty(\Omega) \) and \( \|k\| < 1 \). Necessary and sufficient conditions for the system of equations \( \delta f = k(z)\delta f \) to be locally plentiful are given, and under them a representation of \( k \) also is given.

1. Introduction. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( C^\infty(\Omega) \) denote the space of infinitely differentiable complex valued functions on \( \Omega \). Let \( a \) and \( b \) be in \( C_{(1,0)}^\infty(\Omega) \), the space of \( C^\infty \) differential forms of type \((1, 0)\) on \( \Omega \). Now, consider the \( R \)-linear mapping \( \nu: C^\infty(\Omega) \to C_{(1,0)}^\infty(\Omega) \) defined by \( \nu(f) = \partial f - \bar{f}a - fb \) for \( f \in C^\infty(\Omega) \), where \( \partial \) denotes the operator \( \sum_{j=1}^n \partial_j dz_j \) with \( \partial_j = d/dz_j \). Then, \( \text{Ker } \nu \), the kernel of the map \( \nu \), is an \( R \)-submodule of \( C^\infty(\Omega) \).

Quite recently there has been increasing interest in \( \text{Ker } \nu \), whose elements are called generalized analytic functions of several complex variables (see \([5, 6, 7, 8]\) and references cited in \([7]\)). We call the equation \( \nu(f) = 0 \) the generalized Cauchy-Riemann equation.

Magomedov and Paramodov \([6]\) introduced the idea of the plentifulness of \( \text{Ker } \nu \) to obtain the integrability conditions of the equation \( \nu(f) = 0 \) with \( a = 0 \) on \( \Omega \). When \( \dim_R \text{Ker } \nu \) is infinite on \( \Omega \), \( \text{Ker } \nu \) is said to be plentiful on \( \Omega \). The plentifulness on \( \Omega \) leads to a complex foliation of codimension one of \( \Omega \) determined by the form \( b \). The null sets of generalized analytic functions are leaves of this foliation.

In \([3]\) the author treated generalized analytic functions under the conditions on \( b \) such that a complex foliation of codimension one of \( \Omega \) follows from them.

In this paper we are concerned with the \( R \)-linear mapping \( \alpha: C^\infty(\Omega) \to C_{(1,0)}^\infty(\Omega) \) defined by

\[
\alpha(f) = \sum_{j=1}^n \left\{ \kappa(z) \partial_j f - \partial_j \bar{f} \right\} dz_j, \quad \text{for } f, \kappa \in C^\infty(\Omega).
\]

\( \text{Ker } \alpha \) also is an \( R \)-submodule of \( C^\infty(\Omega) \) as the map \( \nu \). The equation \( \alpha(f) = 0 \) was investigated by S. Hitotumatu \([2]\) and by the author \([4]\). The former used function-theoretic methods and the latter differential equation-theoretical ones.

In \([4]\), given some conditions upon the coefficient \( \kappa \), we discussed properties of elements of \( \text{Ker } \alpha \) (which we call pseudoholomorphic functions with characteristic
similar to those of holomorphic functions and obtained a local representation theorem of such functions.

Now, by using the results in [6, 7] we can obtain necessary and sufficient conditions for Ker $\alpha$ to be plentiful, because the system of elliptic differential equations $\alpha(f) = 0$ can be reduced to the system of type $\nu(g) = 0$ by $f = g + \bar{g}$. However, the methods and results of [6, 7] are not effectual ones to clarify completely structures of Ker $\alpha$.

The purpose of this paper is to investigate relations between $\dim_{\mathbb{R}} \text{Ker} \alpha$ and coefficient $\kappa$, and to give a local representation of $\kappa$ in the case where Ker $\alpha$ is plentiful.

2. Preliminaries and notations. Since Ker $\alpha$ is an $\mathbb{R}$-submodule of $C^\infty(\Omega)$, following Magomedov and Paramodov, when $\dim_{\mathbb{R}} \text{Ker} \alpha$ is infinite, we say that Ker $\alpha$ or the system of differential equations

$$(2.1) \quad \partial_j \bar{f} = \kappa(z) \partial_j f, \quad j = 1, 2, \ldots, n,$$

is plentiful on $\Omega$.

To attain our objective we need a few assumptions on characteristic $\kappa$. First we assume $\|\kappa\| = \sup |\kappa(z)| < 1$.

If $\partial \kappa$ vanishes on an open subset $U$ in $\Omega$, then, considering the restriction of $\alpha$ to $U$ denoted by $\alpha|_U$, we can see that Ker$(\alpha|_U)$ is plentiful [4]. Or, if $\kappa$ vanishes on $U$, then (2.1) is the Cauchy-Riemann equations on $U$. By these reasons we may assume that, for nowhere dense subsets $E_1$ and $E_2$ of $\Omega$,

$$(2.2) \quad \kappa \neq 0 \quad \text{on} \quad \Omega \setminus E_1, \quad \partial \kappa \neq 0 \quad \text{on} \quad \Omega \setminus E_2.$$

Let $C^{(p,q)}(\Omega)$ denote the space of $C^\infty$ differential forms of type $(p, q)$ on $\Omega$.

We shall define the $R$-linear mapping $\alpha^*$ of $C^\infty(\Omega)$ into $C^{(0,1)}(\Omega)$ by

$$\alpha^*(f) = \sum_{j=1}^n \{ \kappa(z) \partial_j f \} \eta_j = \alpha(f).$$

Then, we may regard $\alpha$ and $\alpha^*$ as $R$-linear differential operators of first order on $C^\infty(\Omega)$.

Let $\sigma$ be a vector field on $U$ and $f$ in $C^\infty(U)$. When $\sigma f = 0$ and $\sigma \bar{f} = 0$ on $U$, we say that the vector field $\sigma$ is tangential to $f$. And when, for every $f \in \text{Ker}(\alpha|_U)$, $\sigma$ is tangential to $f$, we say that $\sigma$ is tangential to Ker$(\alpha|_U)$.

To seek vector fields tangential to Ker$(\alpha|_U)$, we need to construct three $C$-linear mappings $\beta$, $\beta$ and $\theta$: $C^\infty(\Omega) \to C^{(p,q)}(\Omega)$ such that their kernels contain Ker $\alpha$ and Ker $\alpha^*$.

Rewriting the map $\alpha$ by using $\partial$, we have $\alpha(f) = \kappa(z) \partial f - \partial \bar{f}$ for $f \in C^\infty(\Omega)$.

Then we have readily the $C$-linear mapping $\partial \alpha$: $C^\infty(\Omega) \to C^{(0,1)}(\Omega)$ defined by

$$\partial \alpha(f) = \partial \kappa \wedge \bar{f} \quad \text{for} \quad f \in C^\infty(\Omega).$$

We put

$$(2.4) \quad \beta = \partial \alpha \quad (= \partial \kappa \wedge \partial).$$

Then we obtain

$$(2.5) \quad \beta(\bar{f}) = \kappa \partial \alpha(f) - \partial \kappa \wedge \alpha(f).$$
We thus define the mapping \( \bar{\beta} : C^\infty(\Omega) \to C^\infty(\Omega) \) as
\[
(2.6) \quad \bar{\beta}(f) = \beta(\bar{f}) \quad \text{for} \ f \in C^\infty(\Omega).
\]
From the definition of \( a^* \) and (2.3)–(2.6) we obtain
\[
(2.7) \quad \text{Ker} \ a^* = \text{Ker} \ a, \quad \text{Ker} \ a \subset \text{Ker} \ \beta = \text{Ker} \ \bar{\beta}.
\]
Lastly, we want to construct a mapping \( \theta \) of \( C^\infty(\Omega) \) into \( C^\infty(\Omega) \). To do this, we need the identity: for \( f \in C^\infty(\Omega) \)
\[
\kappa \{ \bar{\partial} \partial a(f) + \partial \kappa \wedge \partial a^*(f) - \bar{\kappa} \delta \beta(\bar{f}) - \bar{\kappa} \delta \kappa \wedge a(f) \} + \partial \kappa \wedge \partial \bar{\kappa} \wedge a^*(f)
\]
\[
= \kappa (1 - |\kappa|^2) \bar{\partial} \delta \kappa \wedge \bar{\partial} f + \partial \kappa \wedge \partial \bar{\kappa} \wedge \bar{\partial} f.
\]
Then, \( \theta \) is defined as follows:
\[
(2.8) \quad \theta(f) = \kappa (1 - |\kappa|^2) \bar{\partial} \delta \kappa \wedge \bar{\partial} f + \partial \kappa \wedge \partial \bar{\kappa} \wedge \bar{\partial} f \quad \text{for} \ f \in C^\infty(\Omega),
\]
where \( \bar{\delta} = \sum_{j=1}^n \overline{\partial_j} dz_j, \overline{\partial_j} = \partial / \partial \bar{z}_j. \)
The three mappings defined above may be regarded as \( C \)-linear differential operators of first order on \( C^\infty(\Omega) \).
It follows from (2.7) and the above identity that
\[
(2.9) \quad \text{Ker} \ a \subset \text{Ker} \ \theta.
\]
For the purposes of later convenience, we now express (2.4) and (2.8) in terms of coordinates in \( C^n \).
We put
\[
\kappa_i = \partial \kappa, \quad \beta_{ij} = \kappa_i \partial_j - \kappa_j \partial_i,
\]
\[
\gamma_{ijk} = (\overline{\partial_k} \kappa_i) \partial_j - (\overline{\partial_k} \kappa_j) \partial_i,
\]
\[
\theta_{ijk} = \kappa (1 - |\kappa|^2) \gamma_{ijk} + \beta_{ij}(\kappa) \overline{\partial_k}.
\]
From now on, the indices \( i, j \) and \( k \) (with or without subscripts) run over the set \( \{1, 2, \ldots, n\} \) unless specifically stated otherwise.
Then we have
\[
\beta = \partial \kappa \wedge \partial = \sum_{i<j} \{ (\partial_i \kappa) \partial_j - (\partial_j \kappa) \partial_i \} \, dz_i \wedge \, dz_j,
\]
\[
\theta = \sum_{i<j, k} \theta_{ijk} \, dz_i \wedge \, dz_j \wedge \, dz_k.
\]
In the following section we shall prove that on \( \Omega \)
\[
(2.10) \quad \bar{\delta} \partial \kappa \wedge \partial \kappa = 0, \quad \partial \kappa \wedge \partial \bar{\kappa} = 0.
\]
In terms of coordinates of \( C^n \) we rewrite the left sides of (2.10).
\[
\bar{\delta} \partial \kappa \wedge \partial \kappa = \sum_{i<j, k} \gamma_{ijk}(\kappa) \, dz_i \wedge \, dz_j \wedge \, dz_k,
\]
\[
\partial \kappa \wedge \partial \bar{\kappa} = \sum_{i<j} \beta_{ij}(\kappa) \, dz_i \wedge \, dz_j.
3. Necessary conditions for plentifulness. Let \( w \) be a nonconstant pseudoholomorphic function on \( \Omega \). By the unique continuation property for pseudoholomorphic functions [4], we have a nowhere dense subset \( E_3 \) of \( \Omega \) such that \( \partial w \neq 0 \) on \( \Omega \setminus E_3 \). If we put \( E = E_1 \cup E_2 \cup E_3, \kappa \neq 0, \partial \kappa \neq 0 \) and \( \partial w \neq 0 \) on \( \Omega \setminus E \). Since \( \partial \kappa \land \partial w = 0 \) on \( \Omega \), for any \( z \in \Omega \setminus E \) there is a number \( i' \) such that \( \partial_{i'} \kappa \neq 0 \) and \( \partial_{i'} w \neq 0 \). Though we must prove (2.10) about each point of \( \Omega \setminus E \), it is enough to prove it about a specific point. We may assume without loss of generality that if \( 0 \in \Omega \setminus E \), then \( w(0) = 0, \partial_{i'} \kappa(0) \neq 0 \) and \( \partial_{i'} w(0) \neq 0 \).

To prove (2.10) we use the following special change of variables on a small neighborhood \( U \) of the origin

\[
\xi_j = z_j, \quad j = 1, 2, \ldots, n - 1, \quad \xi = w(z).
\]

This is nonsingular because \( w \) satisfies (2.1).

We put \( \partial_{i'} = \partial / \partial \xi_j, \quad \tilde{\partial}_{i'} = \partial / \partial \tilde{\xi}_j, \quad \Delta = \kappa(1 - |\kappa|^2) \) and \( \kappa_{ik} = \tilde{\partial}_{i'} \kappa_{jk} \). Moreover we denote by \( [ \ ] \) the functions into which ones in \( [ \ ] \) are transformed by (3.1). If we note (2.9), i.e. \( \theta_{ijk}(f) = 0 \) for any \( f \in \text{Ker } \alpha \), \( \theta_{ijk} \) are transformed into the following on \( U \):

\[
\begin{align*}
[\theta_{ijk}]'' &= \Delta'' \left\{ \left[ \kappa_{ik} \right]'' \partial_j' - \left[ \kappa_{jk} \right]'' \partial_i' \right\} + \left[ \beta_{ij}(\kappa) \right]'' \tilde{\partial}_k + \left[ \theta_{ijk}(\widetilde{w}) \right]'' \partial_j'' \\
&\quad \text{for } i \neq n, j \neq n, k \neq n, \\
\end{align*}
\]

\[
\begin{align*}
[\theta_{njk}]'' &= \Delta'' \left\{ \left[ \kappa_{nj} \right]'' \partial_j' - \left[ \kappa_{nk} \right]'' \partial_i' \right\} + \left[ \beta_{nj}(\kappa) \right]'' \tilde{\partial}_k + \left[ \theta_{njk}(\widetilde{w}) \right]'' \partial_j'' \\
&\quad \text{for } j \neq n, k \neq n, \\
\end{align*}
\]

\[
\begin{align*}
[\theta_{ijn}]'' &= \Delta'' \left\{ \left[ \kappa_{in} \right]'' \partial_j' - \left[ \kappa_{jn} \right]'' \partial_i' \right\} + \left[ \theta_{ijn}(\widetilde{w}) \right]'' \partial_j'' \\
&\quad \text{for } i \neq n, j \neq n. \\
\end{align*}
\]

**Lemma 1.** If \( \text{Ker } \alpha \) has an element \( W \) linearly independent of \( w \), then the vector fields \( \theta_{ijk} \) are tangential to \( w \).

**Proof.** It is sufficient to show \( \theta_{ijk}(\widetilde{w}) = 0 \). Assume there are a point \( z' \) and numbers \( i', j', k' \) such that \( \theta_{ijk}(\widetilde{w}) \neq 0 \) at \( z' \). We may regard \( z' \) as the origin and, shrinking \( U \) mentioned above if necessary, assume that \( \theta_{ijk}(\widetilde{w}) \neq 0 \) on \( U \). Using the coordinates introduced in (3.1), we obtain, on the image of \( U \) by (3.1), \( \partial_{j'} W'' = 0, \partial_{i'} W'' = 0 \) \((j = 1, \ldots, n - 1 \) and \( \partial_{i'} W'' = 0 \), where we use the relation derived from (3.2), \( \theta_{ijk}(W) = [\theta_{ijk}(w)]'' \partial_{i'} W'' \). Therefore we see \( W'' \) depends only on \( \xi \) and is holomorphic at \( 0 \).

However, since \( \alpha(W) = 0 \), \( [\kappa \partial_n w](\partial_{i'} W'' - \partial_{i'} \widetilde{W}'') = 0 \), and hence \( \partial_{i'} W'' = \partial_{i'} \widetilde{W}'' \). Thus we obtain \( W = aw + b \) on \( U \), where \( a > 0 \) is a constant and \( b \in C \), which contradicts the assumption.

Let \( S \) be any subset of \( \Omega \). The set \( N(S) \) of those vector fields on \( S \) which are tangential to \( w \) is a \( C^\infty(S) \)-submodule of the \( C^\infty(S) \)-module \( M(S) \) consisting of all vector fields on \( S \).

If \( f \in \text{Ker } \alpha \) is nonconstant, nonempty level sets \( \{ z \in \Omega | f(z) = \text{const.} \} \) are \((n - 1)\)-dimensional complex submanifolds except the set of nonordinary points of \( f \) [2].
By virtue of (2.7) every vector field $\beta_j$ is tangential to $\text{Ker} \, \alpha$. If $0 \in \Omega \setminus E$, letting $\beta_i$ denote $\beta_{in}$ $(i = 1, \ldots, n-1)$, we see from the above-mentioned that \{ $\beta_i$, $\bar{\beta}_i$ \} span $N(U)$.

We say $\text{Ker} \, \alpha$ is trivial when it is $C$ itself.

**LEMMA 2.** Under the same assumption as in Lemma 1, (2.10) holds on $\Omega$.

**PROOF.** Lemma 1 shows $\theta_{ijk} \in N(\Omega)$. Assume $0 \in \Omega \setminus E$. Then $\theta_{ijk}$ can be written by linear combinations of $\beta_s$, $\bar{\beta}_s$ $(s = 1, \ldots, n-1)$ with coefficients in $C^\infty(U)$. If $a_{ijk}^s$, $b_{ijk}^s$ denote the coefficients of $\beta_s$, $\bar{\beta}_s$, we have:

For $1 < i < j < n - 1$ and each $k$,

$$\kappa_n a_{ijk}^s = \Delta \kappa_{jk}, \quad \kappa_n a_{ijk}^s = -\Delta \kappa_k, \quad \sum_{s=1}^{n-1} a_{ijk}^s \kappa_s = 0.$$  

(3.3)

For $1 < i < j = n$ and each $k$,

$$a_{ink}^s = 0, \quad 1 < s < n - 1, s \neq i,j,$$

(3.4)

$$\kappa_n a_{ink}^s = \Delta \kappa_{nk}, \quad \kappa_n a_{ink}^s = \Delta \kappa_{jk}.$$

For $1 < k < n - 1$ and each $i,j$,

$$b_{ijk}^s = 0, \quad 1 < s' < n - 1, s' \neq k,$$

(3.5)

$$\bar{\kappa}_k b_{ijk}^s = 0, \quad \bar{\kappa}_k b_{ijk}^s = -\beta_{ijk}(\bar{\kappa}).$$

For $k = n$ and each $i,j$, $\beta_{ijk}(\bar{\kappa}) = 0$ by $\beta_{jn}^s = 0$, $1 < s < n - 1$.

If $k \neq n$, from (3.5) we need to consider the following two cases.

Case 1. For all $k$, $1 < k < n - 1$, $\kappa_k = 0$ on $U$.

Case 2. For some $k'$, $1 < k' < n - 1$ and some point $z' \in U$, $\kappa_k' \neq 0$ at $z'$.

We prove the first part of (2.10) only in Case 1. Let there be an open subset $V$ of $U$ and some $i', j'$ ($i' < j'$) such that $\beta_{i'j'}(\bar{\kappa}) = 0$. Then, $\beta_i = \partial_i, i = 1, \ldots, n - 1$, and so

$$\bar{\kappa}_k = [ \beta_{i'j'}(\bar{\kappa}) ]^{-1} \{ \theta_{ijk} + (\Delta \kappa_{jk} \beta_i) \} \in N(V).$$

Hence $N(V) = M(V)$, which contradicts the nontrivial $\text{Ker} \, \alpha$.

We next prove the second part of (2.10). From (3.3) and (3.4) we have $\gamma_{ijk} = \kappa_{ik} \kappa_j - \kappa_{jk} \kappa_i = 0$ on $U$, which completes the proof.

**COROLLARY.** Let $U$ be an open subset of $\Omega$. If on $U$ either (1) $\partial k \wedge \partial \bar{k} \neq 0$, $\partial \partial k \wedge \partial k = 0$ or (2) $\partial k \wedge \partial \bar{k} = 0$, $\partial \partial k \wedge \partial k \neq 0$, then $\text{Ker} \, \alpha$ is trivial.

**PROOF.** Let $w \in \text{Ker} \, \alpha$ be nonconstant.

Case (1). $\partial \partial k \wedge \partial k = 0$ leads to $\theta(w) = \partial k \wedge \partial \bar{k} \wedge \partial w$. By (2.7) and (2.9), $\partial w = 0$ on $U$, $w$ is constant.

Case (2). $\partial k \wedge \partial \bar{k} = 0$ leads to $\theta(w) = \partial \partial k \wedge \partial w$. Since $\partial k \wedge \partial w = 0$, $\theta(w) = c\partial \partial k \wedge \partial k$ for some function $c \in C^\infty(U)$, and hence $\partial \partial k \wedge \partial k = 0$ on $U$, which contradicts the assumption.

**THEOREM 1.** For the system (2.1) to be plentiful on $\Omega$, it is necessary that the characteristic $\kappa$ fulfill the condition (2.10) on $\Omega$. 


4. Sufficient conditions for plentifulness. We show the local validity of the converse of Theorem 1. As is readily verified, the first half of (2.10) is sufficient for the \((1, 0)-\text{form } \sum \kappa_j dz_j\) to determine a complex foliation of codimension one of \(\Omega\). The converse of this is not always valid (see, e.g. Example (ii) below).

**Lemma 3.** For a function \(\kappa \in C^\infty(\Omega)\) satisfying \(\partial \kappa \neq 0\) and (2.10) on \(\Omega\), there is locally a holomorphic function \(h\) such that \(dh \land \partial \kappa = 0\), \(dh \land \partial h = 0\) and \(dh \neq 0\).

**Proof.** If we put \(\omega = \partial \kappa\), by the first half of (2.10) \(\overline{\partial} \omega = \rho \land \omega\) for a form \(\rho \in C^\infty(\Omega)\). From this \(\overline{\partial} \rho \land \omega = 0\), \(\rho \neq 0\) leads to \(\overline{\partial} \rho = 0\) on \(\Omega\), so that for each point of \(\Omega\) there are a neighborhood \(U\) of that point and a function \(g \in C^\infty(U)\) such that \(\overline{\partial} g = \rho\). Putting \(\tau = \omega \exp(-g)\), we see \(\overline{\partial} \tau = 0\), which shows \(\tau\) is a holomorphic form. By using \(\omega = \tau \exp g\), we have \(\partial g \land \tau + d\tau = 0\), and hence

\[
\tau \land d\tau = 0, \quad \tau \neq 0 \quad \text{on } U.
\]

Let \(H(U)\) denote the algebra of all holomorphic functions on \(U\). We define \(\tau = \sum \tau_j dz_j\) and \(D_{ij} = \tau_j \partial_i - \tau_i \partial_j, \tau_j \in H(U)\). Then, by (4.1) we have a function \(h\) holomorphic on a neighborhood \(V \subset U\) such that \(dh \neq 0\) and \(\tau \land dh = 0\) (i.e. \(D_{ij}\) is tangential to \(h\)). Thus the proof is complete.

**Lemma 4 [4, Theorem 20].** Assume that \(\kappa\) satisfies (2.10) and \(\partial \kappa \neq 0\) on \(\Omega\). Then

(2.1) locally reduced to the equation of one variable

\[
\partial_z F = \overline{K(t)} \partial_z F, \quad |K| < 1,
\]

where \(K(t)\) is defined and of class \(C^\infty(h(V))\).

**Proof.** We have a holomorphic function \(h\) satisfying the conditions of Lemma 3 on an open subset \(V\) of \(\Omega\). We may assume \(h\) is the coordinate function \(z_n\). Since \(\partial \kappa \land dz_n = 0\), \(\partial \kappa \land dz_n = 0\), \(\kappa\) and \(\bar{\kappa}\) are holomorphic in the other coordinates when fixing \(z_n\), and so is a function of \(z_n\) alone. Then \(\kappa = K(z_n)\) and, for any \(f \in \ker \alpha\), \(\partial f \land dz_n = \overline{\partial f} \land dz_n = 0\), so \(f = F(z_n)\). Thus equation (4.2) is obtained.

We now take a disk \(\delta \subset h(V)\). Then we have

**Lemma 5.** Equation (4.2) is plentiful on \(\delta\).

**Proof.** Let \(\Delta_i (i = 1, 2)\) be disks concentric with \(\delta\) such that \(\delta \subset \Delta_1 \subset \Delta_2\). We take a function \(K_1(t) \in C^\infty\) on \(C\) as follows: \(K_1(t)\) equals one on \(\delta\) and zero outside \(\Delta_1\). Besides, it fulfills \(0 < K_1(t) < 1\). Putting \(L(t) = K_1(t)K(t)\), we have the equation

\[
\partial_z G = \overline{L(t)} \partial_z \overline{G}.
\]

Consider the Dirichlet problem of (4.3) with the boundary conditions \(\Re G(t) = g(t)\) on \(\partial \Delta_2\), and \(G(t_0) = 0, t_0 \in \Delta_2\), where \(g(t)\) is a given real valued continuous function on \(\Delta_2\). This problem is solvable [1]. The plentifulness on \(\delta\) of (4.2) is derived from the fact that \(\dim \Re C(\partial \Delta_2)\) is infinite and from the unique continuation property for solutions of equation (4.3). Thus we have the following

**Theorem 2.** If the characteristic \(\kappa\) satisfies (2.10) on \(\Omega\), then \(\ker \alpha\) is locally plentiful.

Using Lemma 3 and the proof of Lemma 4, we have a local representation of \(\kappa\).
Theorem 3. There are locally a holomorphic function $h$ and $K(t) \in \mathbb{C}^\infty(\text{img } h)$, $t = h(z)$, such that $\kappa = K \circ h$, $dh \neq 0$ and $K \neq 0$ if and only if $\kappa$ satisfies (2.10) and $\partial \kappa \neq 0$ on $\Omega$.

Examples. (i) Consider $\kappa = \phi(z) + \overline{\psi(z)}$, where $\phi$ and $\psi$ are holomorphic and $d\phi \wedge d\psi = 0$ on $\Omega$. Then $\partial \kappa \wedge \partial \kappa \neq 0$, even though $\bar{\partial} \partial \kappa \wedge \partial \kappa = 0$, so $\text{Ker } \alpha$ is trivial by the corollary to Lemma 2.

(ii) We consider (2.1) on a small neighborhood $U$ of the origin in $\mathbb{C}^2$ such that $w(z)$ is well defined on $U$ by the following equation:

\[(w + z_1 + z_2)^2 - 2w = 2z_2, \quad w(0) = 0.\]

Putting $\kappa = w(z) + \bar{z}_1 + \bar{z}_2$ (restricting $U$ further if necessary) we can observe that $\text{Ker } \alpha$ is generated only by $w(z)$ (see also [4]). A simple computation shows that $\partial_1 w - \partial_2 w \neq 0$ on $U$, and so it is easy to obtain $\partial \kappa \wedge \partial \kappa = (\partial_1 w - \partial_2 w) \, dz_1 \wedge dz_2 \neq 0$. Moreover we can also show

\[(\bar{\partial} \partial \kappa \wedge \partial \kappa \neq 0 \quad \text{on } U.\]

(iii) We regard the function $w$ defined by (4.4) as $\kappa(z)$. Evidently we have $\partial \kappa \wedge \partial \kappa = 0$ and (4.5) on $U$. By the corollary to Lemma 2 $\text{Ker } \alpha$ is trivial.

(iv) Global plentifulness is not always true, even though $\kappa$ satisfies (2.10) on $\Omega$. The following example shows global plentifulness is valid.

If we take $\kappa = (2/3)(z_1^2 + z_2)$ on $\Omega = \{z \in \mathbb{C}^2: |z_1^2 + z_2| < 1\}$, then we have easily, for any nonnegative integers $m$,

\[w = (z_1^2 + z_2)^m + 0 + (2/3)(z_1^2 + z_2)^{m+2} / (m + 2).\]

It is trivial for $\text{Ker } \alpha$ to be plentiful on $\Omega$.

In conclusion we are in a position to state relations between $\dim R \text{Ker } \alpha$ and $\kappa$.

We define the following notations: $d = \dim R \text{Ker } \alpha$, $\theta_1 = \partial \kappa \wedge \partial \kappa$ and $\theta_2 = \bar{\partial} \partial \kappa \wedge \partial \kappa$.

1. If $d > 3$, $\theta_1 = \theta_2 = 0$ on $\Omega$.
2. If $\theta_1 = \theta_2 = 0$ on $\Omega$, $d = +\infty$ locally.
3. If there is an open subset $U$ of $\Omega$ on which either $\theta_1 \neq 0$, $\theta_2 = 0$ or $\theta_1 = 0$, $\theta_2 \neq 0$, then $d = 1$ (Ker $\alpha$ is trivial).
4. If there is a point of $\Omega$ at which $\theta_1 \neq 0$, $\theta_2 \neq 0$, then $d < 2$.

Acknowledgment. The author expresses his sincere thanks to the referee for his helpful advice, especially for indicating a way to prove Lemma 4 without quoting Theorem 20 [4].

References


Department of Mathematics, Himeji Institute of Technology, Shosha 2167, Himeji, 671-22, Japan