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TOTALLY REAL MINIMAL IMMERSIONS OF n-DIMENSIONAL REAL SPACE FORMS INTO n-DIMENSIONAL COMPLEX SPACE FORMS

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ABSTRACT. n-dimensional totally real minimal submanifolds of constant sectional curvature in n-dimensional complex space forms are totally geodesic or flat.

1. Introduction. B. Y. Chen and K. Ogiue [1] showed that an n-dimensional, totally real, minimal submanifold of constant curvature c in an n-dimensional complex space form is totally geodesic or $c < 0$. On the other hand, [2, Theorem 7] implies that a complete totally real minimal surface of constant sectional curvature in a 2-dimensional complex space form is totally geodesic or flat. We shall prove a generalization of these results.

THEOREM. *Let M be an n-dimensional, totally real, minimal submanifold of constant sectional curvature c, immersed in an n-dimensional complex space form. Then M is totally geodesic or flat* ($c = 0$).

2. Preliminary. We denote by $M^n(4\bar{c})$ an n-dimensional complex space form of constant holomorphic sectional curvature $4\bar{c}$ with complex structure J and metric $\bar{g}$. Let M be an n-dimensional Riemannian manifold of constant sectional curvature c isometrically immersed in $M^n(4\bar{c})$ as a totally real submanifold. We denote by σ the second fundamental form of the immersion

$$\sigma(X, Y) = \bar{\nabla}_X Y - \nabla_X Y,$$

where $\bar{\nabla}$ (resp. ∇) is the covariant differentiation with respect to $\bar{g}$ (resp. the metric g of M).

We put $T = -J\sigma$. Then T is a symmetric tensor field of type (1, 2) on M and it satisfies

$$g(T(X, Y), Z) = g(T(X, Z), Y). \tag{2.1}$$

Moreover, the equations of Gauss, Ricci and Codazzi are given respectively by [1],

$$\begin{aligned}(\bar{c} - c)\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} + g(T(X, Z), T(Y, W))\\ -g(T(X, W), T(Y, Z)) = 0 \quad \text{(the equations of Gauss and Ricci)},\end{aligned} \tag{2.2}$$

$$(\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z) = 0 \quad \text{(the equation of Codazzi)}. \tag{2.3}$$

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3. A lemma. In this section, we prove the following.

LEMMA. *Let T be a symmetric 3-linear map of $R^n \times R^n \times R^n$ into R such that*

$$(3.1)\quad \begin{aligned} &A\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\}+\sum_{m=1}^{n} T(X,Z,f_m)T(Y,W,f_m)\\ &\quad -\sum_{m=1}^{n} T(X,W,f_m)T(Y,Z,f_m)=0 \quad \text{and} \quad A \geqslant 0, \end{aligned}$$

$$(3.2)\quad \sum_{m=1}^{n} T(X,f_m,f_m)=0,$$

where g is the Euclidean metric of R^n and $f_1, \ldots, f_n$ is an orthonormal basis. If we choose an orthonormal basis $e_1, \ldots, e_n$, such that each e_i is a maximum point of the cubic function $T(X, X, X)$, restricted to $\{X \in R^n\colon \|X\| = 1$, and X is orthogonal to $e_1, \ldots, e_{i-1})\}$, then T has the following expression:

$$T(e_a,e_a,e_a)=(n-a)\sqrt{\frac{A}{(n-a+1)}+\cdots+\frac{A}{(n-a+1)\ldots n}},$$

$$T(e_a,e_{i_a},e_{j_a})=-\sqrt{\frac{A}{(n-a+1)}+\cdots+\frac{A}{(n-a+1)\ldots n}}\,\delta_{i_a j_a},$$

where $1 \leqslant a \leqslant n$ and $a \leqslant i_a, j_a \leqslant n$ unless $i_a = j_a = a$.

PROOF. If $A = 0$, the assumption (3.1) and (3.2) imply $T = 0$. Hence we may consider the case $A > 0$. We shall prove this lemma by induction on the dimension of R^n. It is easy to prove that $T(e_1, e_1, X) = 0$ for all X orthogonal to e_1. Since $T(e_1, X, Y)$ is symmetric with respect to X and Y, we can choose an orthonormal basis $f_1(= e_1), f_2, \ldots, f_n$ which satisfies $T(f_1, f_i, f_j) = \lambda_i\delta_{ij}$. Using (3.1) and (3.2), we obtain

$$(3.3)\quad \lambda_1 > 0,$$

$$(3.4)\quad \lambda_1 + \cdots \lambda_n = 0,$$

$$(3.5)\quad A + \lambda_1\lambda_a - (\lambda_a)^2 = 0 \quad \text{for } 1 < a \leqslant n.$$

If $n = 2$, the result follows from (3.3), (3.4) and (3.5).

Assume that the lemma is true for $\leqslant n - 1$ and consider the lemma for R^n. Let $f_1, \ldots, f_n$ be the orthogonal basis chosen above. From (3.5), we must consider two cases.

Case 1. $\lambda_2 = \cdots = \lambda_{p+1}\ (= \mu)$ and $\lambda_{p+2} = \cdots = \lambda_n\ (= \nu)$, where $\mu \neq \nu$ and $1 \leqslant p \leqslant n - 2$.

Case 2. $\lambda_2 = \cdots = \lambda_n\ (= \mu)$.

If Case 1 holds, then, without loss of generality, we may assume $2p \leqslant n - 1$. From (3.4) and (3.5), it follows that

$$\mu^2 = (n-p)A/(p+1), \qquad \nu^2 = (p+1)A/(n-p),$$
$$\lambda_1\mu = (n-2p-1)A/(p+1) \quad \text{and} \quad \lambda_1\nu = -(n-2p-1)A/(n-p).$$

Thus we have $n - 2p - 1 > 0$ and hence, $n > 3$,

$$\mu = \sqrt{(n-p)A/(p+1)}\,, \qquad \gamma = -\sqrt{(p+1)A/(n-p)}\,,$$
$$\lambda_1 = \sqrt{(n-p)A/(p+1)} - \sqrt{(p+1)A/(n-p)}\,.$$

Therefore we use the following convention on the ranges of indices: $a < a' < p + 1 < a'' < n$. Using (3.1), we have $(\lambda_{a'} - \lambda_{a''})T(f_a, f_{a'}, f_{a''}) = 0$, which, together with $\lambda_{a'} - \lambda_{a''} \neq 0$, implies $T(f_a, f_{a'}, f_{a''}) = 0$. Let N (resp. N'; N'') be the linear subspace of R^n spanned by $f_2, \dots, f_n$ (resp. $f_2, \dots, f_{p+1}$; $f_{p+2}, \dots, f_n$). Then we obtain

$$T(X, Y', Z'') = 0, \qquad T(f_1, X', Y') = \sqrt{(n-p)A/(p+1)}\, g(X', Y'),$$
$$T(f_1, X'', Y'') = -\sqrt{(p+1)A/(n-p)}\, g(X'', Y''), \qquad T(f_1, X', Y'') = 0,$$

where $X \in N$, X', $Y' \in N'$ and X'', Y'', $Z'' \in N''$, which, together with (3.1) and (3.2), imply that

$$A(n+p)/(p+1)\{g(X', Z')g(Y', W') - g(X', W')g(Y', Z')\}$$
$$+ \sum_{a'=2}^{p+1} T(X', Z', f_{a'})T(Y', W', f_{a'}) - \sum_{a'=2}^{p+1} T(X', W', f_{a'})T(Y', Z', f_{a'}) = 0,$$
$$\sum_{a'=2}^{p+1} T(X', f_{a'}, f_{a'}) = 0$$

and

$$A(n+1)/(n-p)\{g(X'', Z'')g(Y'', W'') - g(X'', W'')g(Y'', Z'')\}$$
$$+ \sum_{a''=p+2}^{n} T(X'', Z'', f_{a''})T(Y'', W'', f_{a''})$$
$$- \sum_{a''=p+2}^{n} T(X'', W'', f_{a''})T(Y'', Z'', f_{a''}) = 0,$$
$$\sum_{a''=p+2}^{n} T(X'', f_{a''}, f_{a''}) = 0,$$

where X', Y', Z', $W' \in N'$ and X'', Y'', Z'', $W'' \in N''$. Since the dimensions of N' and N'' are less than n, from the assumption we obtain unit vectors $e' \in N'$ and $e'' \in N''$ such that

$$T(e', e', e') = (p-1)\sqrt{A(n+1)/p(p+1)}\,,$$
$$T(e'', e'', e'') = (n-p-2)\sqrt{A(n+1)/(n-p-1)(n-p)}\,.$$

Therefore the definition of e_1 gives

$$\sqrt{A(n-p)/(p+1)} - \sqrt{A(p+1)/(n-p)}$$
$$> \mathrm{Max}\Big\{(p-1)\sqrt{A(n+1)/p(p+1)}\,,$$
$$(n-p-2)\sqrt{A(n+1)/(n-p-1)(n-p)}\Big\},$$

which implies $\sqrt{(n-p)/(p+1)} \geqslant (p-1)\sqrt{(n+1)/p(p+1)}$. We immediately obtain $p = 1$ or 2. This, together with the inequality, induces a contradiction for $n > 3$. It is easy to treat Case 2 by the same argument as Case 1. Q.E.D.

4. Proof of Theorem. Let T be the second fundamental form of the immersion as a symmetric bilinear map $TM \times TM$ into TM. By (2.1), we may consider T as a symmetric 3-linear map of $TM \times TM \times TM$ into R. By (2.2) and the minimality of M, it satisfies (3.1) and (3.2) for $A = \bar{c} - c$. We may assume that M is not totally geodesic, i.e., $A \neq 0$. We easily obtain a local field of orthonormal frames $e_1, \ldots, e_n$ such that the lemma holds. We denote by ω_j^i the Levi-Civita connection with respect to $e_1, \ldots, e_n$. Using (2.3), we have

$$-\sqrt{\frac{A}{n}}\sum_{i=1}^{n}\omega_a^i(e_1)e_i - \sum_{i=1}^{n}\omega_a^i(e_1)T(e_i, e_1) - \sum_{i=1}^{n}\omega_a^i(e_1)T(e_a, e_i)$$
$$-(n-1)\sqrt{\frac{A}{n}}\sum_{i=1}^{n}\omega_1^i(e_a)e_i + 2\sum_{i=1}^{n}\omega_1^i(e_a)T(e_i, e_1) = 0, \quad \text{for all } a \neq 1.$$

Taking the innerproduct of it and e_1, we obtain $\omega_a^1(e_1) = 0$. This, together with the innerproduct of the above and e_b ($b \neq 1$), implies $\omega_1^a(e_b) = 0$. As a result, e_1 is a parallel vector field on M. Thus M is flat. Q.E.D.

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