

CALCULATING THE FUNDAMENTAL GROUP OF AN ORBIT SPACE

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ABSTRACT. Suppose G acts effectively as a group of homeomorphisms of the connected, locally path connected, simply connected, locally compact metric space X . Let \bar{G} denote the closure of G in $\text{Homeo}(X)$, and N the smallest normal subgroup of \bar{G} which contains the path component of the identity of \bar{G} and all those elements of \bar{G} which have fixed points. We show that $\pi_1(X/G)$ is isomorphic to \bar{G}/N subject to a weak path lifting assumption for the projection $X \rightarrow X/\bar{G}$.

Given a topological space X together with a group G of homeomorphisms of X , what can we say about the fundamental group of the orbit space X/G ? Results for simplicial and discontinuous groups have been given in [1] and [2]. The object of this note is to produce a theorem which can deal with both discontinuous and continuous actions.

We shall assume that X is a connected, locally path connected, locally compact metric space. Let G be a group of homeomorphisms of X which acts effectively on X , so that we can think of G as a subgroup of the group $\text{Homeo}(X)$ of all homeomorphisms of X endowed with the compact open topology.

Under very reasonable hypotheses (see conditions A, B, C below) the answer to our question is as follows. Let \bar{G} denote the closure of G in $\text{Homeo}(X)$, and let N be the smallest normal subgroup of \bar{G} which contains the path component of the identity of \bar{G} and all those elements of \bar{G} which have fixed points. Then if X is simply connected the fundamental group of X/G is isomorphic to the quotient group \bar{G}/N .

Suppose X fails to be simply connected but has a universal covering space \tilde{X} . Each homeomorphism $g: X \rightarrow X$ lifts to a homeomorphism of \tilde{X} , and any two lifts of the same g differ by a covering transformation. Therefore we have an action of an extension of $\pi_1(X)$ by G on \tilde{X} whose orbit space is homeomorphic to X/G , and we can apply our result in this setting. Details of the group extension and of its action on \tilde{X} can be found in [5] and [3].

Here are some examples to illustrate a variety of situations in which the result can be used.

EXAMPLE 1. Take $S^2 \times \mathbf{R}$ for X and $S^1 \times \mathbf{Z}$ for G , the action being as follows. The circle acts on S^2 by rotation leaving the north and south poles fixed, and acts trivially on \mathbf{R} . The generator of \mathbf{Z} reflects S^2 in the equator and translates \mathbf{R} along

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one unit. Then $G = \bar{G}$, $N = S^1 \times \{0\}$ and the fundamental group of the orbit space (which is easily seen to be the Möbius strip) is \mathbf{Z} .

EXAMPLE 2. Consider the action of $G = \text{PSL}(2, \mathbf{Z})$ on the upper half plane U as a group of linear fractional transformations $z \mapsto (az + b)/(cz + d)$. Then G is a closed subgroup of $\text{Homeo}(U)$, and is generated by the elements $z \mapsto -\frac{1}{z}$, $z \mapsto 1 - \frac{1}{z}$ both of which have fixed points. Therefore the orbit space U/G is simply connected. In fact the region

$$\left\{ z \in U \mid |z| \geq 1, |\text{Re}(z)| \leq \frac{1}{2} \right\}$$

is a fundamental region for the action of G , and looking at the way in which its sides are identified shows that U/G is homeomorphic to a punctured sphere.

EXAMPLE 3. Consider the group of rationals \mathbf{Q} acting on the real line \mathbf{R} by addition. Taking the closure of \mathbf{Q} in $\text{Homeo}(\mathbf{R})$ gives a copy of \mathbf{R} acting on itself by translation. Since \mathbf{R} is path connected, \mathbf{R}/\mathbf{Q} must be simply connected.

EXAMPLE 4. Let G be a compact Lie group acting effectively on a simply connected space X . Assume that either G is connected, or that X^G (those points fixed by all elements of G) is nonempty. In both cases $G = \bar{G} = N$ and the orbit space X/G must be simply connected.

EXAMPLE 5. Consider an irrational flow on the torus T . More precisely, let \mathbf{R} act on $T \cong S^1 \times S^1$ as follows: the real number r sends

$$(e^{2\pi i x}, e^{2\pi i y}) \text{ to } (e^{2\pi i(x+r)}, e^{2\pi i(y+r\sqrt{2})}).$$

This action lifts to an action of $\pi_1(T) \times \mathbf{R}$ on \mathbf{R}^2 which has the same orbit space, namely $(m, n, r) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{R}$ sends (x, y) to $(x + m + r, y + n + r\sqrt{2})$. One easily checks that the closure of this group of homeomorphisms of \mathbf{R}^2 is precisely the group of all translations of \mathbf{R}^2 . Since this is a path connected group, the orbit space T/\mathbf{R} is simply connected.

EXAMPLE 6. Let F be a finitely generated free group and X the graph of F relative to a minimal set S of generators. (That is to say X has a vertex for each element of F , and an edge joins vertices g and h if and only if $h^{-1}g$ is an element of S or the inverse of an element of S .) Note that X is simply connected because F is free. The action of F on itself by left multiplication induces a free simplicial action of F on X . If G is any subgroup of F we have an induced action of G on X and the orbit space X/G is a one-dimensional simplicial complex. The fundamental group of this orbit space must therefore be free. But in terms of our result $G = \bar{G}$ and N is the trivial subgroup, so this fundamental group is just G and we recapture Nielsen's theorem that any subgroup of F is free.

The conditions we shall need are listed below.

- A. The projection $X \rightarrow X/\bar{G}$ has the path lifting property up to homotopy.
- B. Given points $x, x' \in X$ plus a neighborhood V of x in X , if $x' \in \bar{G}x$ then $x' \in GV$.
- C. The group \bar{G}/N acts discontinuously on X/N .

(The technical terms mentioned above are defined as follows. A map $f: X \rightarrow Y$ has the path lifting property up to homotopy if given a path $\alpha: I \rightarrow Y$, and a point $p \in f^{-1}\alpha(0)$, we can find a path $\beta: I \rightarrow X$ such that $\beta(0) = p$, $f\beta(1) = \alpha(1)$ and

$f\beta \simeq \alpha \text{ rel}\{0, 1\}$. A group G acts discontinuously on a space X if each point x of X has a neighborhood V such that $gV \cap V$ is empty unless g fixes x .)

PROPOSITION 1. *If condition B is satisfied then X/G and X/\bar{G} have the same homotopy type.*

PROOF. Let $\pi: X \rightarrow X/G$ and $\phi: X/G \rightarrow X/\bar{G}$ denote the natural projections. Define $\psi: X/\bar{G} \rightarrow X/G$ as follows; given $z \in X/\bar{G}$ choose a point $y \in X/G$ which satisfies $\phi(y) = z$ and set $\psi(z) = y$. Then $\phi\psi$ is the identity map of X/\bar{G} .

We need to show $\psi\phi$ homotopic to the identity map of X/G . To this end define $F: X/G \times I \rightarrow X/G$ by

$$F(y, t) = \begin{cases} \psi\phi(y), & 0 \leq t < 1, \\ y, & t = 1. \end{cases}$$

If F is continuous we are finished. Now given an open set U of X/G we have

$$F^{-1}(U) = [\phi^{-1}\psi^{-1}(U) \times [0, 1)] \cup [U \times \{1\}].$$

Clearly this is open in $X/G \times I$ if $U \subseteq \phi^{-1}\psi^{-1}(U)$, in other words if $\phi(U) \subseteq \psi^{-1}(U)$.

Suppose $z \in \phi(U)$, say $\phi(y) = z$ where $y \in U$. We need to check that $\psi(z)$ lies in U . Let $\psi(z) = y'$, then $\phi(y') = \phi(y) = z$. Choose $x, x' \in X$ with images y, y' respectively in X/G . The set $V = \pi^{-1}(U)$ is an open neighborhood of x in X , and $x' \in \bar{G}x$ since x and x' both map to z in X/\bar{G} . By hypothesis we must have $x' \in GV$. Therefore $\pi(x')$ lies in $\pi(GV)$, that is to say $\psi(z)$ lies in U as required.

COROLLARY 2. *If G is a group of isometries of X , then X/G and X/\bar{G} have the same homotopy type.*

PROOF. Simply check condition B for a group of isometries. Suppose we have $x, x' \in X$ such that $x' \in \bar{G}x$, and let V be a neighborhood of x in X . Let ϵ denote the distance from x to $X - V$, and choose $g \in G$ such that $g(x)$ is within ϵ of x' . Then, since g is an isometry, $x' \in gV$ as required.

Condition A is the only one which is hard to check, so we list several situations where it is satisfied.

PROPOSITION 3. *The projection $X \rightarrow X/\bar{G}$ has the path lifting property up to homotopy if any one of the following holds.*

- (a) G acts simplicially on a triangulation of X .
- (b) The action of G on X is discontinuous, and the stabiliser of any point is finite.
- (c) \bar{G} is a compact Lie group.
- (d) \bar{G} is a locally compact Lie group and acts properly on X .
- (e) X/\bar{G} is semilocally simply connected.

PROOF. (a), (b), (c), (d) can be found in [1], [2], [3], [4], respectively, and in these cases one can actually lift the given path, rather than just a path which is homotopic to it.

Case (e). We use an argument due to Smale [6], though our hypotheses are less restrictive. Let $\alpha: I \rightarrow X/\bar{G}$ be the given path, let $y_0 = \alpha(0)$ and suppose $x_0 \in \pi^{-1}(y_0)$. For each point y of X/\bar{G} choose a neighborhood Uy such that loops in Uy are null homotopic in X/\bar{G} . We can do this since X/\bar{G} is semilocally simply connected. Given $x \in \pi^{-1}(y)$ use the local path connectedness of X , and the continuity of π , to find a path connected neighborhood $Py(x)$ of x such that $\pi(Py(x)) \subseteq Uy$. If $x' \in \pi^{-1}(y)$ also, set $Py(x') = gPy(x)$ where g is an element of \bar{G} which sends x to x' . Let $Vy = \pi(Py(x))$ and note that Vy is a neighborhood of y in X/\bar{G} since π is an open map.

Divide up I into N subintervals of equal length, taking N sufficiently large so that $1/N$ is a Lebesgue number for the open covering $\{\alpha^{-1}(Vy) | y \in X/\bar{G}\}$ of I . Set $I_1 = \{t \in I | 0 \leq t \leq 1/N\}$ and note that $\alpha(I_1)$ must be contained in Vy for some point $y \in X/\bar{G}$. Therefore $y_0 \in Vy$ and $x_0 \in Py(x)$ for some $x \in \pi^{-1}(y)$. Now $Py(x)$ is path connected, so we can join x_0 to some point of $Py(x) \cap \pi^{-1}(\alpha(1/N))$ by a path $\gamma: I \rightarrow Py(x)$. Define $\beta: I_1 \rightarrow X$ by $\beta(t) = \gamma(Nt)$. Then $\beta(0) = x_0$, $\pi\beta(1/N) = \alpha(1/N)$, and $\pi\beta \simeq \alpha|_{I_1} \text{ rel } \{0, 1/N\}$, since $\pi\beta$ and $\alpha|_{I_1}$ have the same end points and both lie in Uy . This defines our lift (up to homotopy) β over I_1 ; the remaining subintervals are dealt with in a similar manner.

THEOREM 4. *If X is simply connected and if conditions A, B and C are satisfied, then $\pi_1(X/G)$ is isomorphic to \bar{G}/N .*

PROOF. We shall show that X/N is simply connected. Assume this is done and note that \bar{G}/N acts freely on X/N , because N contains all the elements of \bar{G} which have fixed points, and that it acts discontinuously by hypothesis. Therefore the projection $X/N \rightarrow X/\bar{G}$ is a covering map and we deduce $\pi_1(X/\bar{G}) \cong \bar{G}/N$. The theorem now follows from Proposition 1.

Choose a base point $p \in X$ and let $q = \pi(p)$, where π now stands for the projection from X to X/N . Define $\phi: N \rightarrow \pi_1(X/N, q)$ as follows. Given an element $g \in N$, join p to $g(p)$ in X by a path γ and set

$$\phi(g) = \langle \pi \circ \gamma \rangle.$$

Notice that the choice of γ , amongst all paths joining p to $g(p)$, is irrelevant since X is simply connected. This function ϕ is a homomorphism, for given $g_1, g_2 \in N$, and having chosen γ_1 joining p to $g_1(p)$ and γ_2 joining p to $g_2(p)$, use $\gamma_1 \cdot g_1(\gamma_2)$ to join p to $g_1g_2(p)$. Then

$$\phi(g_1g_2) = \langle \pi \circ (\gamma_1 g_1(\gamma_2)) \rangle = \langle \pi \circ \gamma_1 \rangle \langle \pi \circ \gamma_2 \rangle = \phi(g_1)\phi(g_2).$$

The kernel of ϕ is all of N . For if g lies in the path component of the identity of \bar{G} , join e to g by a path $\{g_t | 0 \leq t \leq 1\}$ in \bar{G} . Then $\{g_t(p)\}$ joins p to $g(p)$ in X and projects to a single point in X/N . Therefore g lies in the kernel of ϕ . Now suppose g is an element of \bar{G} which fixes some point of X , say $g(x) = x$. Join p to x by a path γ , and use $\gamma(g\gamma)^{-1}$ to join p to $g(p)$. This path projects to a null homotopic loop in X/N , so again g lies in the kernel of ϕ .

It only remains to show that ϕ is onto. Given an element of $\pi_1(X/N, q)$ represent it by a loop α based at q . Now $X \rightarrow X/\bar{G}$ has the path lifting property up

to homotopy and, since $X/N \rightarrow X/\bar{G}$ is a covering map, so has $X \rightarrow X/N$. Therefore we can find a path β in X which begins at p and which satisfies $\pi\beta(1) = q$, $\pi\beta \simeq \alpha \text{ rel}\{0, 1\}$. Since $\beta(1) \in \pi^{-1}(q)$ there is an element $g \in N$ such that $g(p) = \beta(1)$, and by construction $\phi(g) = \langle \pi \circ \beta \rangle = \langle \alpha \rangle$. This completes the proof.

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