PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

GENERATORS OF $H^*(MSO; Z_2)$ AS A MODULE OVER THE STEENROD ALGEBRA, AND THE ORIENTED COBORDISM RING

STAVROS PAPASTAVRIDIS

ABSTRACT. In this paper we will describe a minimal set of A-generators of $H^*(MSO; Z_2)$ (where A is the mod-2 Steenrod Algebra). The description is very much analogous to R. Thom's description of generators for $H^*(MO; Z_2)$ (see [7]). As a corollary, we give simple cohomological criteria for a manifold to be indecomposable in the oriented cobordism. Our proof relies on work of D. J. Pengelley (see [5]).[1]

0. Statement of results. In order to describe our results, we need some terminology.

All homology and cohomology groups of this paper will have Z_2 coefficients. Ω will be the oriented cobordism ring.

The cohomology of BO will be identified, in the well-known way, with the subalgebra of $Z_2[t_1, t_2, \ldots, t_N]$, which consists of all symmetric polynomials (each time the index N will be big enough for our purposes).

1. DEFINITION. We will call a partition a finite sequence of positive integers $\omega = (a_1, a_2, \ldots, a_k)$, so that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$. We will call the degree of ω the integer $|\omega| = a_1 + a_2 + \cdots + a_k$. We call the length of ω the number of terms which appear in ω, i.e. $l(\omega) = k$.

If ω is a partition, then $s(\omega)$ is the well-known element of $H^{|\omega|}(BO)$, i.e.

$$s(\omega) = \sum t_{i_1}^{a_1} t_{i_2}^{a_2} \cdots t_{i_k}^{a_k}.$$

It is well known that the $s(\omega)$'s form a Z_2-basis for $H^*(BO)$, and the elements of the form $s(\omega) \cdot U$ (where $U \in H^0(MO)$ is the Thom class) constitute a Z_2-basis for $H^*(MO)$ (see [2]).

If M is any closed, compact and C^∞ manifold, then $s(\omega)(M) \in Z_2$ is the corresponding normal characteristic number of M.

Let $I\colon MSO \to MO$ be the obvious map.

2. DEFINITION. We define P to be the set of all partitions ω which satisfy all the following conditions.

(a) No number of the form $(2^i - 1)$, where $i \geqslant 1$, is included in ω.

Received by the editors December 15, 1980 and, in revised form, April 13, 1981.
1980 *Mathematics Subject Classification*. Primary 55R40, 57R75; Secondary 55S10, 57R20.
Key words and phrases. Oriented cobordism, cohomology of MSO.

[1]The original proof was lengthier and elementary. After the original version of the paper was written, I was informed of Pengelley's unpublished work, which could be used to shorten the argument.

0002-9939/82/0000-0426/$02.50

(b) A number of the form 2^i, where $i \geq 1$, appears always an even number of times in the partition ω. (*Remark*. The number zero is even.)

We define P_1 to be the subset of P which consists of all partitions of the form $(2a_1, 2a_1, 2a_2, 2a_2, \ldots, 2a_k, 2a_k)$ where $0 < a_1 \leq a_2 \leq \cdots \leq a_k$.

We define P_2 to be the subset of $(P - P_1)$ which consists of all partitions of the form $(a_1, a_2, \ldots, a_k)$ or of the form $(a_1, a_2, \ldots, a_k, 2b_1, 2b_1, \ldots, 2b_m, 2b_m)$, where a_k is an odd number.

Our main result is the following

3. THEOREM. *The set of elements* $\{I^*(s(\omega) \cdot U)\colon \omega \in P_1 \cup P_2\}$ *is a minimal set of generators for the A-module* $H^*(MSO)$. *The only relations are*

$$\mathrm{Sq}^1(I^*(s(\omega) \cdot U)) = 0, \quad \textit{where } \omega \in P_1.$$

4. DEFINITION. Let P_3 be the set of all partitions $(2a, 2a)$, where $a > 0$. Let P_4 be the subset of P_2 which consists of all partitions of the form $(a_1, a_2, \ldots, a_k)$ where a_k is an odd number and the $a_1, a_2, \ldots, a_{k-1}$'s are unequal even numbers.

The following two theorems are corollaries of Theorem 3.

5. THEOREM. *Let M be an oriented manifold whose oriented cobordism class belongs to the torsion part of* Ω. *The manifold M is indecomposable in* Ω *if and only if there is a partition* $\omega \in P_4$ *so that* $s(\omega)(M) \neq 0$.

The corresponding condition for the free part of Ω is well known (see [**8**, p. 293]).

6. THEOREM. *Let* $\{M_{4k}\colon k \geq 1\}$ *be a collection of oriented manifolds which form a minimal set of generators of the free part of* Ω. *Let* $\{M_\omega\colon \omega \in P_4\}$ *be a collection of oriented manifolds so that* dim $M_\omega = |\omega|$. *Then, the collection of manifolds*

$$\{M_{4k}, M_\omega\colon k \geq 1, \omega \in P_4\},$$

is a minimal set of generators for Ω *if and only if the matrix* $\|s(\omega')(M_\omega)\|$, *where* $\omega, \omega' \in P_4$, *is invertible.*

1. The A_*-comodule structure of $H_*(MSO)$. The main result of this section is Theorem 12, which is a corollary of D. Pengelley's work (see Theorem 8) and provides certain information concerning the A_*-comodule structure of $H_*(MSO)$. (*Remark*. A_* is the dual of the mod-2 Steenrod Algebra.)

Let $\{x(\omega)\colon \omega$ is a partition$\}$ be the basis of $H_*(MO)$, which is dual to the basis $\{s(\omega) \cdot U\colon \omega$ is a partition$\}$ of $H^*(MO)$, and let $x(\omega)$ be the dual of $s(\omega) \cdot U$.

The following theorem is well known.

7. THEOREM. *Let* $x_i = x((i))$, *where* $i > 0$, *and let* $\omega = (a_1, a_2, \ldots, a_k)$ *be a partition. Then* $H_*(MO)$ *is a polynomial algebra so that*

$$H_*(MO) = Z_2[x_1, x_2, \ldots, x_n, \ldots].$$

Furthermore, we have $x(\omega) = x_{a_1}x_{a_2} \cdots x_{a_k}$.

PROOF. See, for example, [**1**].

Let $\bar{\xi}_i \in A_{*(2^i-1)}$ be the Hopf Algebra conjugate of Milnor's generators $\xi_i \in A_{*(2^i-1)}$.

8. THEOREM (D. J. PENGELLEY). *There is a sequence of elements $y_n \in H_n(MO)$, where $n \geqslant 2$, so that*

$$I_*\big(H_*(MSO)\big) = Z_2[\,y_2, y_3, \ldots, y_n, \ldots\,].$$

If $n \neq 2^i$, then y_n is indecomposable in $H_*(MO)$. If $n = 2^i$, where $i > 1$, then there is an indecomposable element $z_{n/2} \in H_{n/2}(MO)$, so that $y_n = (z_{n/2})^2$. Furthermore, if $\mu_*: H_*(MO) \to A_* \otimes H_*(MO)$ is the obvious coaction map, then we have

$$\text{(a)} \qquad \mu_*(y_n) = \begin{cases} \bar{\xi}_1^2 \otimes 1 + 1 \otimes y_2, & \text{if } n = 2, \\ 1 \otimes (z_{n/2})^2, & \text{if } n = 2^i \text{ and } i \geqslant 2, \\ \Sigma_{j=0}^i \bar{\xi}_j \otimes y_{2^{i-j}-1}^{2^j}, & \text{if } n = 2^i - 1 \text{ and } i > 2, \\ 1 \otimes y_n + \bar{\xi}_1 \otimes y_{n-1}, & \text{if } n = 2k \text{ and } k \neq 2^i, \\ 1 \otimes y_n, & \text{otherwise.} \end{cases}$$

PROOF. See [5].

9. DEFINITION. Let ω_1, ω_2 be two partitions. We say that ω_1 is bigger than ω_2 if and only if at least one of the following two conditions is satisfied:

(a) $l(\omega_1) > l(\omega_2)$.

(b) $l(\omega_1) = l(\omega_2)$ and $|\omega_1| < |\omega_2|$.

This relation of "bigger" is clearly transitive but it is not a total ordering.

10. DEFINITION. Let $\omega_1, \omega_2, \ldots, \omega_k$ be k distinct partitions and let ω be another partition. Let $a, a_1, \ldots, a_k$ be nonzero elements of A_*. We say that the element

$$a_1 \otimes x(\omega_1) + a_2 \otimes x(\omega_2) + \cdots + a_k \otimes x(\omega_k)$$

of $A_* \otimes H_*(MO)$ is bigger than $a \otimes x(\omega)$ if and only if all the partitions $\omega_1, \omega_2, \ldots, \omega_k$ are bigger than ω.

11. DEFINITION. Let $x, y \in A_* \otimes H_*(MO)$. We define the symbol $x \prec y$ to mean that the element $(x - y)$ is bigger than x, or that $(x - y) = 0$.

REMARK. We caution the reader about the fact that the relation $\prec$ is *not* defined for arbitrary elements of $A_* \otimes H_*(MO)$.

Our next result is a corollary of Theorem 8.

12. THEOREM. *We have*

(a) $\bar{\xi}_1^2 \otimes 1 \prec \mu_*(y_2)$.

(b) *If $n = 2^i$ and $i > 2$, then*

$$1 \otimes x_{(n/2)}^2 \prec \mu_*(y_n).$$

(c) *If $n = 2^i - 1$ and $i \geqslant 2$, then*

$$\bar{\xi}_i \otimes 1 \prec \mu_*(y_n).$$

(d) *If $n \neq 2^i, 2^i - 1$, for $i > 0$, then*

$$i \otimes x_n \prec \mu_*(y_n).$$

2. The Steenrod Algebra. In this section we will describe two well-known lemmas about the Steenrod Algebra, which will be used in the sequel.

13. LEMMA. *Let B be the subspace of A generated by the elements* $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\ldots\mathrm{Sq}^{i_k}$ *where $i_{t-1} \geqslant 2i_t$, $k \geqslant t \geqslant 2$ and $i_k \geqslant 2$. Then A is the direct sum of the subspaces B and $B \cdot \mathrm{Sq}^1$. Furthermore $A \cdot \mathrm{Sq}^1 = B \cdot \mathrm{Sq}^1$.*

PROOF. See [4, p. 7–8].

14. LEMMA. *The subspace of A_* which is the annihilator of $A \cdot \mathrm{Sq}^1$ is the polynomial subalgebra of A_* generated by $\bar{\xi}_1^2$, $\bar{\xi}_i$ for $i \geqslant 2$.*

PROOF. Let Sq^R, where $R = (r_1, r_2, \ldots)$, be the well-known element of the Milnor s basis of A (see [3]). Milnor proves that

$$\mathrm{Sq}^1\mathrm{Sq}^R = (r_1 + 1)\mathrm{Sq}^{r_1+1, r_2, \cdots}.$$

This implies that the elements $\bar{\xi}_1^2$, $\bar{\xi}_i$ for $i \geqslant 2$, belong to the annihilator of $A\ \mathrm{Sq}^1$. The rest of the proof follows from the dimensions of the Z_2-spaces B, $B \cdot \mathrm{Sq}^1$, $Z_2[\bar{\xi}_1^2, \bar{\xi}_2, \bar{\xi}_3, \ldots]$.

3. Proof of Theorem 3. In this section we will prove Theorem 3, but first we need some preparation.

15. DEFINITION. Let X be a subset of a vector space. Then $\langle X \rangle$ is the subspace spanned by X.

Let C be a set of partitions. Then we define $s(C) = \{s(\omega)\colon \omega \in C\}$.

Let $\omega = (a_1, a_2, \ldots, a_k)$ be a partition. Then we define $y(\omega) = y_{a_1}y_{a_2}\cdots y_{a_k}$.

16. PROPOSITION. *The restriction of the map $I^*\mu$,*

$$I^*\mu\colon B \otimes \langle s(P) \cdot U \rangle \to H^*(MSO)$$

is an isomorphism.

(*Remark*. For the definition of B, see Lemma 13. For the definition of P, see Definition 2.)

PROOF. First we observe that the graded spaces $B \otimes \langle s(P) \cdot U \rangle$ and $H^*(MSO)$ have the same Z_2-dimensions in each degree. This follows easily from the definition of B, P and the cohomology of MSO.

So it is enough to prove that the restriction of the map $I^*\mu$ is a monomorphism. We argue by contradiction. Let us assume that there are k distinct partitions $\omega_1, \omega_2, \ldots, \omega_k$ of P and nonzero elements of B, $a_1, a_2, \ldots, a_k$, so that

$$I^*\mu(a_1 \otimes s(\omega_1)U + a_2 \otimes s(\omega_2)U + \cdots + a_k \otimes s(\omega_k)U) = 0.$$

Among the partitions $\omega_1, \omega_2, \ldots, \omega_k$, there is at least one which is maximal in the relation bigger. Let us suppose that ω_1 is such a maximal partition. Let ω be the partition that we get by substituting in ω_1 every occurrence of $(\ldots, 2^i, 2^i, \ldots)$ by $(\ldots, 2^{i+1}, \ldots)$. Then, by Theorem 12, we have $1 \otimes x(\omega_1) \prec \mu_*(y(\omega))$. Besides, there is an element b belonging to the annihilator of $A \cdot \mathrm{Sq}^1 = B \cdot \mathrm{Sq}^1$, so that $\langle a_1, b \rangle = 1$. Furthermore, by Lemma 14, the element b can be chosen to be a product of $\bar{\xi}_1^2$, $\bar{\xi}_i$'s where $i \geqslant 2$. So, by Theorem 12, there is a partition ω', consisting entirely of 2, $(2^i - 1)$'s where $i \geqslant 2$, so that

$$b \otimes 1 \prec \mu_*(y(\omega')).$$

Finally, by Theorem 8, there is an element $z \in H_*(MSO)$ so that $I_*(z) = y(\omega') \cdot y(\omega)$.

Combining all the above we get $I_*(z) = y(\omega')y(\omega_1)$. Combining the above, we have

$$\begin{aligned}
&\langle I^*\mu(a_1 \otimes s(\omega_1) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U), z\rangle \\
&\quad = \langle a_1 \otimes s(\omega) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U, \mu_* I_*(z)\rangle \\
&\quad = \langle a_1 \otimes s(\omega_1) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U, \mu_*(y(\omega')y(\omega))\rangle \\
&\quad = \langle a_1 \otimes s(\omega_1) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U, b \otimes x(\omega_1)\rangle \\
&\quad = \langle a_1 \otimes s(\omega_1) \cdot U, b \otimes x(\omega_1)\rangle = 1 \neq 0
\end{aligned}$$

which contradicts our assumption.

17. LEMMA. *Let $\omega \in (P - (P_1 \cup P_2))$ be a partition. Then there is a partition $\omega_0 \in P_2$ so that $s(\omega) - \mathrm{Sq}^1 s(\omega_0) = \Sigma_i s(\omega_i)$ where the ω_i's belong to P_2.*

PROOF. Let $\omega = (a_1, \ldots, a_m, 2b_1, 2b_1, \ldots, 2b_k, 2b_k)$ where a_m is an even positive integer and $a_{m-1} < a_m$. Then we define $\omega_0 = (a_1, \ldots, a_{m-1}, a_m - 1, 2b_1, 2b_1, \ldots, 2b_k, 2b_k)$. Clearly $\omega_0 \in P_2$. The assertion of the lemma follows easily.

18. PROPOSITION. *Let R be the subalgebra of A generated by the element Sq^1. Then the Z_2-space $I^*(\langle s(P - P_1) \cdot U\rangle)$ is a free R-module and the set $I^*(s(P_2) \cdot U)$ is a free basis.*

PROOF. The previous lemma says that the set $I^*(s(P_2) \cdot U)$ is a set of R-generators for the R-module $I^*(\langle s(P - P_1) \cdot U\rangle)$. (*Remark.* Note that $\mathrm{Sq}^1 I^*(s(\omega) \cdot U) = I^*(\mathrm{Sq}^1(s(\omega)) \cdot U)$). Next, it is not difficult to observe that the number of partitions of $(P - P_1 \cup P_2)$ of degree m equals the number of partitions of P_2 of degree $(m - 1)$. This implies that the set of elements $\{s(\omega) \cdot U, \mathrm{Sq}^1(s(\omega) \cdot U)\}$ for $\omega \in P_2$ is Z_2-independent. That ends the proof.

PROOF OF THEOREM 3. It follows easily from the results of this section.

4. Proof of Theorems 5 and 6. In this final section, we will complete the proofs of Theorems 5 and 6, but first we will need some preparation.

19. LEMMA. *Let ω be a partition consisting entirely of even numbers, so that at least one of them appears an odd number of times in ω. If M is an orientable manifold, then*

$$s(\omega)(M) = 0.$$

PROOF. Let $\omega = (a_1, a_2, \ldots, a_m, \ldots, a_k)$, so that $a_{m-1} < a_m$ and the number a_m appears in ω an odd number of times. Let $\omega_0 = (a_1, a_2, \ldots, a_m - 1, \ldots, a_k)$.

Clearly

$$\mathrm{Sq}^1(I^*(s(\omega_0)\cdot U)) = I^*(s(\omega)\cdot U).$$

Now the proof follows without charge.

20. COROLLARY. *Let $\omega \in P_3 \cup P_4$ and let M be an oriented manifold which is decomposable in Ω. Then*

$$s(\omega)(M) = 0.$$

21. DEFINITION. Let $\omega \in P_1 \cup P_2$. Then N_ω is defined to be an oriented manifold, so that $s(\omega)(N_\omega) \neq 0$ and $s(\omega')(N_\omega) = 0$ for all $\omega' \in P_1 \cup P_2$ and $\omega' \neq \omega$. The existence of such manifolds is guaranteed by Theorem 3.

22. PROPOSITION. *The family of N_ω's, where $\omega \in P_3 \cup P_4$, is a minimal set of algebra generators for $\Omega \otimes Z_2$.*

PROOF. By the previous corollary, the cobordism classes of N_ω's are linearly independent in $(\Omega \otimes Z_2)/(\text{decomposable})$. On the other hand, these manifolds are as numerous as Wall's generators of Ω (see [8, p. 309]). So, they must generate $\Omega \otimes Z_2$.

23. COROLLARY. *Let $\omega \in P_3$. Then the manifold N_ω of Definition* 21 *can be chosen to be a polynomial generator of the torsion free part of Ω.*

Now the proof of Theorems 5 and 6 follows without difficulty from Proposition 22 and Corollary 23.

BIBLIOGRAPHY

1. A. Liulevicius, *A proof of Thom's theorem*, Comment. Math. Helv. **37** (1962–63), 121–131.

2. ______, *Homology comodules*, Trans. Amer. Math. Soc. **134** (1968), 375–382.

3. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171.

4. N. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies, no. 50, Princeton Univ. Press, Princeton, N. J., 1962.

5. D. J. Pengelley, *The homology of MSO and MSU as A* comodule algebras, and the cobordism ring* (to appear).

6. R. Stong, *Notes on cobordism theory*, Princeton Univ. Press, Princeton, N. J., 1968.

7. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

8. C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PATRAS, PATRAS, GREECE