EXTENSION OF A RESULT OF BEACHY AND BLAIR

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Abstract. Let \( a : R \to R \) be an automorphism of a ring \( R \). If \( R \) is an \( a \)-reduced ring in which every faithful left ideal is cofaithful, then the same is true for the \( a \)-twisted power series ring \( R^*[x] \).

Introduction. A module \( _RM \) is said to be cofaithful if there exists a finite number of elements \( m_1, \ldots, m_k \) in \( M \) with \( \cap_{i=1}^k \text{Ann}_R(m_i) = 0 \). Writing \( M^k \) for the direct sum of \( k \) copies of \( M \), it is clear that \( M \) is cofaithful if and only if there exists an exact sequence \( 0 \to R \to M^k \) in \( R\text{-mod} \), where \( k \) is some integer \( \geq 1 \). A result of Beachy and Blair [1] asserts that if \( R \) is a commutative ring in which every faithful ideal is cofaithful, then the same is true for the polynomial ring \( R[x] \). The object of this note is to extend the validity of this result to skew power series rings under suitable assumptions on the ring \( R \).

Throughout this note \( R \) will denote a ring with identity and all the modules considered will be unitary left modules.

1. \( a \)-reduced rings. Let \( a : R \to R \) be a ring endomorphism satisfying \((1)a = 1\). We write \( a^n \) for \((a)a \) whenever \( a \in R \). The twisted (or skew) power series ring corresponding to the endomorphism \( a \) will have as its elements the usual formal power series \( \sum_{i\geq 0} x^ia_i \) with \( a_i \in R \) \((x^0 = 1 \) by convention) with addition defined in the usual way, but multiplication given by \( x^ix^j = x^{i+j} \) and \( a \cdot x = xa^n \) for \( i \geq 0 \), \( j \geq 0 \) and \( a \in R \). Then \( a \cdot x^r = x^ra^{nr} \) where \( a^r : R \to R \) is the \( r \)th power of \( a \). We will fix an endomorphism \( a \) and denote the twisted power series ring corresponding to \( a \) by \( R^*[x] \).

A ring \( R \) is said to be reduced if there are no nonzero nilpotent elements in \( R \). If \( a, b \) are elements in a reduced ring, it is easily seen that \( ab = 0 \iff ba = 0 \).

Definition 1.1. We say that \( R \) is an \( a \)-reduced if \( R \) is reduced and further satisfies the condition \( ab = 0 \iff a^nb = 0 \iff ab^a = 0 \) for any \( a, b \) in \( R \).

Proposition 1.2. Let \( R \) be an \( a \)-reduced. Let \( f = \sum_{i\geq 0} x^ia_i \), \( g = \sum_{j\geq 0} x^jb_j \) be in \( R^*[x] \). Then \( fg = 0 \) in \( R^*[x] \) if and only if \( a_ib_j = 0 \) for all \( i \geq 0, j \geq 0 \).
Proof. Let $R$ be $\alpha$-reduced and $a_i b_j = 0$ for all $i, j$. For any $k > 0$ the coefficient of $x^k$ in $fg$ is $\sum_{i+j=k; i>0, j>0} a_i^\alpha b_j$. Since $R$ is $\alpha$-reduced, from $a_i b_j = 0$ we immediately get $a_i^\alpha b_j = 0, a_i^\alpha b_j = 0, \ldots, a_i^\alpha b_j = 0$ for all $r > 0$. In particular $a_i^\alpha b_j = 0$. Hence $fg = 0$.

Conversely assume $fg = 0$ in $R[[x]]$. Then for any integer $k > 0$ we get

$$\sum_{i+j=k; i>0, j>0} a_i^\alpha b_j = 0. \quad (S_0)$$

We will refer to the equation $\sum_{i+j=k; i>0, j>0} a_i^\alpha b_j = 0$ as the $k$th equation of the system $(S_0)$. The 0th equation of $(S_0)$ yields $a_0 b_0 = 0$. The first equation of $(S_0)$ is

$$a_0^\alpha b_1 + a_1 b_0 = 0. \quad (A_0)$$

Multiplying this on the left by $b_0$ we get

$$b_0 a_0^\alpha b_1 + b_0 a_1 b_0 = 0. \quad (B_0)$$

Since $R$ is $\alpha$-reduced, from $a_0 b_0 = 0$ we get $b_0 a_0 = 0, a_0^\alpha b_0 = 0 = a_0^\alpha a_0^\alpha, b_0^\alpha a_0^\alpha = 0 = b_0 a_0^\alpha$ for all $r > 1$. In particular $b_0 a_0^\alpha = 0$. Now, $(B_0)$ yields $b_0 a_1 b_0 = 0$, which in turn yields $a_1 b_0 a_1 b_0 = 0$. Since $R$ is reduced, this implies that $a_1 b_0 = 0$.

Assume inductively that we have proved that $a_i b_0 = 0$ for $0 < i < l$ (with $l > 1$). The $(l + 1)$st equation in the system $(S_0)$ is

$$a_0^\alpha a_l b_{l+1} + a_l^\alpha b_l + \cdots + a_1^\alpha b_1 + a_{l+1} b_0 = 0. \quad (C_0)$$

Multiplying $(C_0)$ on the left by $b_0$ we get

$$b_0 a_0^\alpha a_l b_{l+1} + b_0 a_l^\alpha b_l + \cdots + b_0 a_1^\alpha b_1 + b_0 a_{l+1} b_0 = 0. \quad (D_0)$$

From $a_i b_0 = 0$ for $0 < i < l$ and the $\alpha$-reducibility of $R$ we get $b_0 a_l^\alpha = 0$ for $0 < i < l$ and all $r > 0$. Now, equation $(D_0)$ yields $b_0 a_{l+1} b_0 = 0$. Hence $a_{l+1} b_0 a_{l+1} b_0 = 0$. Since $R$ is reduced, we get $a_{l+1} b_0 = 0$.

It follows that $a_l b_0 = 0$ for all $l > 0$.

Substituting $(E_0)$ in $(S_0)$, we get for every integer $k > 1$ the system of equations

$$\sum_{i+j=k; i>0, j>1} a_i^\alpha b_j = 0. \quad (S_1)$$

We will refer to the equation $\sum_{i+j=k; i>0, j>1} a_i^\alpha b_j = 0$ as the $k$th equation of the system $(S_1)$. The first equation of $(S_1)$ yields $a_0^\alpha b_1 = 0$. The $\alpha$-reducibility of $R$ now yields $a_0^\alpha b_1 = 0 = b_1 a_0$ and $b_1 a_0^\alpha = 0$ for any $r > 1$. The second equation of the system $(S_1)$ is

$$a_0^\alpha b_2 + a_1^\alpha b_1 = 0. \quad (A_1)$$

Multiplying on the left by $b_1$ we get

$$b_1 a_0^\alpha b_2 + b_1 a_1^\alpha b_1 = 0. \quad (B_1)$$

Using $b_1 a_0^\alpha = 0$ for any $r > 1$ we see from $(B_1)$ that $b_1 a_1^\alpha b_1 = 0$, which in turn yields $a_1^\alpha b_1 a_1^\alpha b_1 = 0$. The reduced nature of $R$ now yields $a_1^\alpha b_1 = 0$. The $\alpha$-reduced nature of $R$ now yields $a_1 b_1 = 0$. 

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Assume inductively that we have proved that \( a_ib_i = 0 \) for \( 0 < i < l \) (with \( l > 1 \)). The \( \alpha \)-reducibility of \( R \) then yields \( b_ia^\alpha = 0 \) for \( 0 < i < l \) and \( r > 0 \). The \((l + 2)\)nd equation of the system \((S_1)\) is

\[
(C_1) \quad a_0^{\alpha^{l+2}}b_{l+2} + a_1^{\alpha^{l+1}}b_1 + \cdots + a_{l+1}a_1b_1 = 0.
\]

Multiplying \((C_1)\) on the left by \( b_1 \) we get

\[
(D_1) \quad b_1a_0^{\alpha^{l+2}}b_{l+2} + \cdots + b_1a_{l+1}a_1b_1 = 0.
\]

Using \( b_1a_i^\alpha = 0 \) for \( 0 < i < l \) in \((D_1)\) we get \( b_1a_{l+1}a_1b_1 = 0 \). This in turn implies \( a_{l+1}a_1b_1b_1b_1 = 0 \) and the reduced nature of \( R \) implies \( a_{l+1}a_1b_1 = 0 \). The \( \alpha \)-reduced nature of \( R \) now yields \( a_{l+1}b_1 = 0 \).

\[(E_1) \quad \text{It follows that } a_i b_1 = 0 \text{ for all } i > 0.
\]

Using \((E_1)\), the system \((S_1)\) yields the system of equations

\[
(S_2) \quad \sum_{i+j=k} a_i^b b_j = 0 \text{ for any } k > 2.
\]

Proceeding as before and using \((S_2)\) we can show that

\[(E_2) \quad a_i b_2 = 0 \text{ for all } i > 0.
\]

Repeating this procedure we see that \( a_i b_j = 0 \) for all \( i, j \).

2. The main result. We will now prove the main result of this paper.

**Theorem 2.1.** Let \( \alpha : R \to R \) be a ring automorphism of \( R \) with \((1)\alpha = 1 \). Let \( R \) be \( \alpha \)-reduced and \( R^*[\![x]\!] \) the \( \alpha \)-twisted power series ring over \( R \). If every faithful left ideal in \( R \) is cofaithful, then every faithful left ideal in \( R^*[\![x]\!] \) is cofaithful.

**Proof.** Let \( I \) be any faithful left ideal of \( R^*[\![x]\!] \). Let \( I_0 = \{ a \in R \mid \text{there exists some } f \in I \text{ with } a \text{ as one of the coefficients occurring in } f \} \). Let \( a, b \) be elements of \( I_0 \). Let \( a = \text{coefficient of } x^i \text{ in } f \) with \( f \in I \) and \( b = \text{coefficient of } x^j \) in \( g \) with \( g \in I \). Then \( a + b = \text{coefficient of } x^{i+j} \text{ in } x^if + x^g \) and \( x^if + x^g \in I \). Hence \( a + b \in I_0 \).

Since \( \alpha \) is an automorphism, \( \alpha^{-1} \) exists. For any \( r \in R \), the coefficient of \( x^i \) in \( r^{\alpha^{-1}} \cdot f \) is precisely \( ra \). Thus \( a \in I_0 \), \( r \in R \Rightarrow ra \in I_0 \). This proves that \( I_0 \) is a left ideal in \( R \).

Let \( r \in \text{Ann}_R(I_0) \). Then for any \( f \in I \), the coefficient of \( x^i \) in \( r^{\alpha^{-1}} \cdot f = r \cdot \text{coefficient of } x^i \text{ in } f \) is \( 0 \). Hence \( r^{\alpha^{-1}} \cdot f = 0 \) for any \( f \in I \). Since \( I \) is faithful in \( R^*[\![x]\!] \), we see that \( r^{\alpha^{-1}} = 0 \). Hence \( r = 0 \). This proves that \( I_0 \) is faithful in \( R \). Hence there exist finitely many elements \( a_1, a_2, \ldots, a_k \) in \( I_0 \) with \( \text{Ann}_R(a_1, \ldots, a_k) = 0 \).

Let \( a_i = \text{coefficient of } x^{i\alpha} \text{ in } f_j \in I \). Let \( g = \sum_{i \geq 0} x^i \lambda_i \) be any element of \( \text{Ann}_R^*[\![x]\!] (\hat{f}_1, \ldots, \hat{f}_k) \). From Proposition 1.2 we see that \( \lambda_i a_j = 0 \) for all \( i > 0 \) and \( 1 < j < k \). Since \( \text{Ann}_R^*[\![x]\!] (\hat{a}_1, \ldots, \hat{a}_k) = 0 \) we get \( \lambda_i = 0 \) for all \( i > 0 \) and hence \( g = 0 \). This proves that \( \text{Ann}_R^*[\![x]\!] (\hat{f}_1, \ldots, \hat{f}_k) = 0 \), thereby showing that \( I \) is cofaithful in \( R^*[\![x]\!] \)-mod.
COROLLARY 2.2. If $R$ is a commutative reduced ring in which every faithful ideal is cofaithful, then the same is true in the ordinary power series ring $R[[x]]$.

PROOF. For a commutative reduced ring $R$, the conclusion of Proposition 1.2 is true. Hence the proof of Theorem 2.1 goes through when $R$ is commutative.

REFERENCES


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