EXTENSION OF A RESULT OF BEACHY AND BLAIR

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Abstract. Let \( \alpha: R \rightarrow R \) be an automorphism of a ring \( R \). If \( R \) is an \( \alpha \)-reduced ring in which every faithful left ideal is cofaithful, then the same is true for the \( \alpha \)-twisted power series ring \( R^*[[x]] \).

Introduction. A module \( R M \) is said to be cofaithful if there exists a finite number of elements \( m_1, \ldots, m_k \) in \( M \) with \( \cap_{i=1}^k \text{Ann}_R(m_i) = 0 \). Writing \( M^k \) for the direct sum of \( k \) copies of \( M \), it is clear that \( M \) is cofaithful if and only if there exists an exact sequence \( 0 \rightarrow R \rightarrow M^k \) in \( \text{R-mod} \), where \( k \) is some integer \( \geq 1 \). A result of Beachy and Blair [1] asserts that if \( R \) is a commutative ring in which every faithful ideal is cofaithful, then the same is true for the polynomial ring \( R[x] \). The object of this note is to extend the validity of this result to skew power series rings under suitable assumptions on the ring \( R \).

Throughout this note \( R \) will denote a ring with identity and all the modules considered will be unitary left modules.

1. \( \alpha \)-reduced rings. Let \( \alpha: R \rightarrow R \) be a ring endomorphism satisfying \( (1)\alpha = 1 \). We write \( a^\alpha \) for \( (a)\alpha \) whenever \( a \in R \). The twisted (or skew) power series ring corresponding to the endomorphism \( \alpha \) will have as its elements the usual formal power series \( \sum_{i \geq 0} x^i a_i \) with \( a_i \in R \) \( (x^0 = 1 \) by convention) with addition defined in the usual way, but multiplication given by \( x^i x^j = x^{i+j} \) and \( a \cdot x = x a^\alpha \) for \( i \geq 0, j \geq 0 \) and \( a \in R \). Then \( a \cdot x^r = x^r a^\alpha \) where \( a^r: R \rightarrow R \) is the \( r \)th power of \( a \). We will fix an endomorphism \( \alpha \) and denote the twisted power series ring corresponding to \( \alpha \) by \( R^*[[x]] \).

A ring \( R \) is said to be reduced if there are no nonzero nilpotent elements in \( R \). If \( a, b \) are elements in a reduced ring, it is easily seen that \( ab = 0 \Leftrightarrow ba = 0 \).

Definition 1.1. We say that \( R \) is \( \alpha \)-reduced if \( R \) is reduced and further satisfies the condition \( ab = 0 \Leftrightarrow a^\alpha b = 0 \Leftrightarrow ab^\alpha = 0 \) for any \( a, b \) in \( R \).

Proposition 1.2. Let \( R \) be \( \alpha \)-reduced. Let \( f = \sum_{i \geq 0} x^i a_i, g = \sum_{i \geq 0} x^i b_i \) be in \( R^*[[x]] \). Then \( fg = 0 \) in \( R^*[[x]] \) if and only if \( a_ib_j = 0 \) for all \( i \geq 0, j \geq 0 \).
Proof. Let $R$ be $\alpha$-reduced and $a_i b_j = 0$ for all $i, j$. For any $k > 0$ the coefficient of $x^k$ in $fg$ is $\sum_{i+j=k; i>0, j>0} a_i^* b_j$. Since $R$ is $\alpha$-reduced, from $a_i b_j = 0$ we immediately get $a_i^* b_j = 0$, $a_i^* b_j = 0$, $\ldots$, $a_i^* b_j = 0$ for all $r > 0$. In particular $a_i^* b_j = 0$. Hence $fg = 0$.

Conversely assume $fg = 0$ in $R[[x]]$. Then for any integer $k > 0$ we get

$$a_i^* b_j = 0.$$  

We will refer to the equation $\sum_{i+j=k; i>0, j>0} a_i^* b_j = 0$ as the $k$th equation of the system (S0). The 0th equation of (S0) yields $a_0^* b_0 = 0$. The first equation of (S0) is

$$a_0^* b_1 + a_1^* b_0 = 0.$$  

Multiplying this on the left by $b_0$ we get

$$b_0 a_0^* b_1 + b_0 a_1^* b_0 = 0.$$  

Since $R$ is $\alpha$-reduced, from $a_0^* b_0 = 0$ we get $b_0 a_0^* = 0$, $b_0 a_0^* = 0 = b_0 a_i^*$ for all $r > 1$. In particular $b_0 a_i^* = 0$. Now, (B0) yields $b_0 a_i^* b_0 = 0$, which in turn yields $a_i^* b_0 = 0$. Since $R$ is reduced, this implies that $a_i^* b_0 = 0$.

Assume inductively that we have proved that $a_i^* b_0 = 0$ for $0 < i < l$ (with $l > 1$). The $(l + 1)$st equation in the system (S0) is

$$a_0^* b_{l+1} + a_1^* b_l + \cdots + a_l^* b_1 + a_{l+1}^* b_0 = 0.$$  

Multiplying (C0) on the left by $b_0$ we get

$$b_0 a_0^* b_{l+1} + b_0 a_1^* b_l + \cdots + b_0 a_l^* b_1 + b_0 a_{l+1}^* b_0 = 0.$$  

From $a_0^* b_0 = 0$ for $0 < i < l$ and the $\alpha$-reducibility of $R$ we get $b_0 a_i^* = 0$ for $0 < i < l$ and all $r > 0$. Now, equation (D0) yields $b_0 a_i^* b_0 = 0$. Hence $a_i^* b_0 = 0$. Since $R$ is reduced, we get $a_i^* b_0 = 0$.

We will refer to the equation $\sum_{i+j=k; i>0, j>1} a_i^* b_j = 0$ as the $k$th equation of the system (S1). The first equation of (S1) yields $a_0^* b_1 = 0$. The $\alpha$-reducibility of $R$ now yields $a_0^* b_1 = 0 = b_1 a_0^*$ and $b_1 a_0^* = 0$ for any $r > 1$. The second equation of the system (S1) is

$$a_0^* b_2 + a_1^* b_1 = 0.$$  

Multiplying on the left by $b_1$ we get

$$b_1 a_0^* b_2 + b_1 a_1^* b_1 = 0.$$  

Using $b_1 a_0^* = 0$ for any $r > 1$ we see from (B1) that $b_1 a_1^* b_1 = 0$, which in turn yields $a_1^* b_1 a_1^* b_1 = 0$. The reduced nature of $R$ now yields $a_1^* b_1 = 0$. The $\alpha$-reduced nature of $R$ now yields $a_1^* b_1 = 0$. 

Assume inductively that we have proved that \( a_ib_i = 0 \) for \( 0 < i < l \) (with \( l > 1 \)). The \( \alpha \)-reducibility of \( R \) then yields \( b_ia_i = 0 \) for \( 0 < i < l \) and \( r > 0 \). The \((l + 2)\)nd equation of the system \((S_l)\) is
\[
(a_i^{i+2}b_{i+2} + a_i^{i+1}b_i + \cdots + a_i^1b_1) = 0.
\]
Multiplying \((C_i)\) on the left by \( b_i \) we get
\[
b_i(a_i^{i+2}b_{i+2} + a_i^{i+1}b_i + \cdots + a_i^1b_1) = 0.
\]
Using \( b_ia_i = 0 \) for \( 0 < i < l \) in \((D_i)\) we get \( b_ia_i^{i+1}b_1 = 0 \). This in turn implies \( a_i^{i+1}b_1a_i^{i+1}b_1 = 0 \) and the reduced nature of \( R \) implies \( a_i^{i+1}b_1 = 0 \). The \( \alpha \)-reduced nature of \( R \) now yields \( a_i^{i+1}b_1 = 0 \).

It follows that \( a_ib_i = 0 \) for all \( i > 0 \).

Using \((E_i)\), the system \((S_i)\) yields the system of equations
\[
\sum_{i+j=k, i>0, j>2} a_i^{i+j}b_j = 0 \quad \text{for any } k > 2.
\]
Proceeding as before and using \((S_j)\) we can show that
\[
a_ib_j = 0 \quad \text{for all } i > 0.
\]
Repeating this procedure we see that \( a_ib_j = 0 \) for all \( i, j \).

2. The main result. We will now prove the main result of this paper.

**Theorem 2.1.** Let \( \alpha : R \to R \) be a ring automorphism of \( R \) with \((1)\alpha = 1 \). Let \( R \) be \( \alpha \)-reduced and \( R[[x]] \) the \( \alpha \)-twisted power series ring over \( R \). If every faithful left ideal in \( R \) is cofaithful, then every faithful left ideal in \( R[[x]] \) is cofaithful.

**Proof.** Let \( I \) be any faithful left ideal of \( R[[x]] \). Let \( I_0 = \{a \in R \mid \text{there exists some } f \in I \text{ with } a \text{ as one of the coefficients occurring in } f\} \). Let \( a, b \) be elements of \( I_0 \). Let \( a = \text{coefficient of } x^i \text{ in } f \text{ with } f \in I \) and \( b = \text{coefficient of } x^j \text{ in } g \text{ with } g \in I \). Then \( a + b = \text{coefficient of } x^{i+j} \text{ in } x^if + x^jg \text{ and } x^if + x^jg \in I \). Hence \( a + b \in I_0 \).

Since \( \alpha \) is an automorphism, \( \alpha^{-1} \) exists. For any \( r \in R \), the coefficient of \( x^i \) in \( r^{\alpha^{-1}} \cdot f \) is precisely \( ra \). Thus \( a \in I_0, r \in R \Rightarrow ra \in I_0 \). This proves that \( I_0 \) is a left ideal in \( R \).

Let \( r \in \text{Ann}_R(I_0) \). Then for any \( f \in I \), the coefficient of \( x^i \) in \( r^{\alpha^{-1}} \cdot f \) is \( r \cdot (\text{coefficient of } x^i \text{ in } f) = 0 \). Hence \( r^{\alpha^{-1}} \cdot f = 0 \) for any \( f \in I \). Since \( I \) is faithful in \( R[[x]] \), we see that \( r^{\alpha^{-1}} = 0 \). Hence \( r = 0 \). This proves that \( I_0 \) is faithful in \( R \). Hence there exist finitely many elements \( a_1, a_2, \ldots, a_k \) in \( I_0 \) with \( \text{Ann}_R(a_1, \ldots, a_k) = 0 \).

Let \( a_i = \text{coefficient of } x^{\mu_i} \text{ in } f_i \in I \). Let \( g = \sum_{i \geq 0} x^i \lambda_i \) be any element of \( \text{Ann}_{R[[x]]}(f_1, \ldots, f_k) \). From Proposition 1.2 we see that \( \lambda_i a_j = 0 \) for all \( i > 0 \) and \( 1 < j < k \). Since \( \text{Ann}_R(a_1, \ldots, a_k) = 0 \) we get \( \lambda_i = 0 \) for all \( i > 0 \) and hence \( g = 0 \). This proves that \( \text{Ann}_{R[[x]]}(f_1, \ldots, f_k) = 0 \), thereby showing that \( I \) is cofaithful in \( R[[x]] \)-mod.
Corollary 2.2. If $R$ is a commutative reduced ring in which every faithful ideal is cofaithful, then the same is true in the ordinary power series ring $R[[x]]$.

Proof. For a commutative reduced ring $R$, the conclusion of Proposition 1.2 is true. Hence the proof of Theorem 2.1 goes through when $R$ is commutative.

References