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ON ENTIRE FUNCTIONS OF AFFINE LINEAGE

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ABSTRACT. In this paper we provide a characterization of entire functions whose zero set is a union of real hyperplanes.

The purpose of this note is to establish a new property of entire functions of several complex variables whose zero sets are a union of hyperplanes. Our interest in this subject arose from questions concerning limits of multivariate B-splines and it is our intention to present this relationship elsewhere **[2]**.

We will say that an entire function $f(z)$, $z \in \mathbf{C}^s$ ($f \not\equiv 0$) has *affine lineage* if its zero set $\{z | f(z) = 0, z \in \mathbf{C}^s\}$ consists of a union of hyperplanes $H = \{z | z \cdot \bar{\zeta} = t, z \in \mathbf{C}^s\}$ where $\zeta \in \mathbf{C}^s - \{0\}$, $t \in \mathbf{R}$, and $z \cdot \bar{\zeta} = z_1\bar{\zeta}_1 + \cdots + z_s\bar{\zeta}_s$, $z = (z_1, \ldots, z_s)$, $\zeta = (\zeta_1, \ldots, \zeta_s)$. When all the hyperplanes in the zero set of f are determined by real vectors, $\zeta \in \mathbf{R}^s$, we say that f has *real affine lineage*. The reason for this terminology comes from a theorem of Motzkin and Schoenberg **[1]**, which characterizes functions of affine lineage as limits of products of affine functions, $a(z) = z \cdot \zeta - t$.

Our main theorem below gives a completely different criterion for a function to have real affine lineage. In the sense described below the requirement of having real affine lineage is shown to be intrinsically *univariate* in nature.

THEOREM. *Let f be an entire function on $\mathbf{C}^s$. Then f has real affine lineage if and only if, for every $x, y \in \mathbf{R}^s$, $f(x + zy)$, $z \in \mathbf{C}$, has only real zeros or is identically zero.*

Thus, in particular, if $P(x)$ is a *polynomial* on $\mathbf{C}^s$ such that for all $x, y \in \mathbf{R}^s$ the univariate polynomial $P(x + zy)$ has only real zeros then P has real affine lineage. Hence according to the Weierstrass factorization theorem from **[1]**, P admits a *global* factorization as a product of affine functions,

$$P(x) = \tau(y_1 \cdot x - t_1) \cdots (y_n \cdot x - t_n), \qquad y_j \in \mathbf{R}^s, t_j \in \mathbf{R}, \tau \in \mathbf{C}.$$

PROOF OF THEOREM. Suppose f has real affine lineage and $f(x + zy)$ has a complex zero, say $z_0 \in \mathbf{C}$, for $x, y \in \mathbf{R}^s$. Then there is a real hyperplane $H = \{z | \zeta \cdot z - t = 0\}$, $\zeta \in \mathbf{R}^s$, $t \in \mathbf{R}$, passing through $z_0 = x + z_0 y$ and lying entirely in the zero set of f. Since $\operatorname{Im} z_0 \neq 0$ it immediately follows that $x + ty$ is in H for all

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$t \in \mathbf{R}$. Hence $f(x + zy) \equiv 0$ and so the necessity of our condition is verified. The sufficiency of our characterization will be proved next.

First, let us observe that it suffices to assume f is real on the real axis, since otherwise we can apply our argument below to $\overline{f(\bar{z})}f(z)$. Now, suppose $f(z_0) = 0$ and $z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbf{R}^s$. We will find a real hyperplane in the zero set of f passing through z_0. To this end, observe that $f(x_0 + zy_0) \equiv 0$ for $z \in \mathbf{C}$ due to the hypothesis on f. Thus $f(x_0) = 0$ and so f admits a factorization $f = p_1 \cdots p_m \Omega$ (in the ring of functions which are analytic in a neighborhood of x_0) where each p_j is a (real) irreducible distinguished polynomial and Ω is a unit. Let us assume for definiteness that $p = p_1$ vanishes on $x_0 + ty_0$, for t in some real neighborhood of the origin. Thus $\nabla p(x_0 + ty_0) \cdot y_0 = 0$ for all t in this neighborhood. Moreover, since p is irreducible it is easy to see that for some t_0 near zero $\nabla p(x_0 + t_0 y_0) \neq 0$. Let us introduce $h(z) = p(z + w_0)$ where $w_0 = x_0 + t_0 y_0$. Thus $h(0) = 0$ and $\nabla h(0) \neq 0$. We will now prove the following property of h. All real vectors $y \in \mathbf{R}^s$ satisfying $\nabla h(0) \cdot y = 0$ are zeros of h, i.e., $h(y) = 0$. Assuming this fact for the moment, it quickly follows that $H = \{z | \nabla p(w_0) \cdot (z - w_0) = 0\}$ is the real hyperplane which we seek. To see this, observe that if $\nabla p(w_0) \cdot (z - w_0) = 0$ we have $\nabla h(0) \cdot (\operatorname{Re} z + t \operatorname{Im} z - w_0) = 0$, all $t \in \mathbf{R}$. Hence $h(\operatorname{Re} z + t \operatorname{Im} z - w_0) = p(\operatorname{Re} z + t \operatorname{Im} z) = 0$ and consequently $f(z) = 0$. In the same manner, we observe that $z_0 \in H$. Thus it remains to prove our assertion concerning h, that is, every real vector y in the domain of h satisfying $\nabla h(0) \cdot y = 0$ is a zero of h.

For ease of notation, we set $\zeta = \nabla h(0)$ and suppose to the contrary that there is a $y \in \mathbf{R}^s$ with $\zeta \cdot y = 0$ while $h(y) \neq 0$. We define $F(t, z) = h(t\zeta + zy)$ and observe that $F(0, 0) = 0$ and $F(0, z) = h(zy) \not\equiv 0$. Therefore the Weierstrass preparation theorem implies that there is an integer $m \geqslant 1$ and a neighborhood U_2 of the origin in $\mathbf{C}^2$ such that on U_2

$$F(t, z) = \left(z^m + A_1(t)z^{m-1} + \cdots + A_{m-1}(t)z + A_m(t)\right) \cdot \Omega(t, z)$$

where each A_j is analytic in a neighborhood U_1 of the origin in $\mathbf{C}$, $A_j(0) = 0$ and $\Omega(0, 0) \neq 0$. Since

$$0 < \|\zeta\|^2 = F_1(0, 0) = A_m'(0)\Omega(0, 0)$$

we have $A_m(t) = dt + \cdots$ where $d \neq 0$. We choose a sign $\delta = \pm 1$ so that $\operatorname{Re} \delta d > 0$. Let $d' = \delta d$ and observe that $F(\delta t^m, tz) = t^m Q(t, z) P(t, z)$ where $Q(t, z) = z^m + B_1(t)z^{m-1} + \cdots + B_{m-1}(t)z + B_m(t)$, with each B_j analytic in U_1, $B_m(0) = d'$, $B_j(0) = 0, j \leqslant m - 1$, and $P(0, 0) \neq 0$. Notice that $m > 2$ because $F_2(0, 0) = \zeta \cdot y = 0$. Consequently, since $Q(0, z) = z^m + d'$ we conclude there exist a $\tau \in \mathbf{R}$, $z_0 \in \mathbf{C} - \mathbf{R}$ with $F(\delta\tau^m, \tau z_0) = 0$. Thus along the line $\delta\tau^m\zeta + \tau z y$, h has a *complex* zero at $z = z_0$. This contradiction completes the proof of our assertion concerning h and also the theorem.

Further discussion of entire functions having affine lineage can be found in [3].

References

1. T. S. Motzkin and I. J. Schoenberg, *On lineal entire functions of n complex variables*, Proc. Amer. Math. Soc. 3 (1952), 517–526.

2. W. Dahmen and C. A. Micchelli, *On limits of multivariate B-splines*, J. Analyse Math. (to appear).

3. ______, *On entire functions of affine lineage*, Report No. 8324, IBM Research Center, Yorktown Heights, N. Y., 1980.

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