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# OSCILLATION THEOREMS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DAMPED TERM

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ABSTRACT. Some new integral criteria for the oscillation of the nonlinear second order differential equation with damped term $y''(t) + p(t)y'(t) + q(t)f(y(t)) = 0$ are given.

**1. Introduction.** Consider the linear differential equation

$$y''(t) + a(t)y(t) = 0 \tag{1}$$

where $a(t) \in C[t_0, \infty)$. By the well-known theorem of Wintner [**12**]

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} du \int_{t_0}^{u} a(s)\, ds = \infty \tag{2}$$

is sufficient for equation (1) to be oscillatory even when $a(t)$ is not assumed positive. Hartman [**5**] proved that the limit cannot be replaced by the upper limit in condition (2). In [**6**], Kamenev extended Wintner's result by using the $n$th primitive

$$A_n(t) = \frac{1}{n!}\int_{t_0}^{t} (t-u)^{n-1} a(u)\, du$$

of the coefficient $a(t)$ for some integer $n \geqslant 3$.

Let $R$ be the set of all real numbers. Considering the nonlinear differential equation

$$y''(t) + q(t)f(y(t)) = 0 \tag{3}$$

where $q \in C[t_0, \infty)$, $f \in C(R)$, $yf(y) > 0$ for $y \neq 0$ and $f'(y) \geqslant 0$ for all $y \in R$. Under the assumption that $q(t)$ is eventually nonnegative, Waltman [**11**] proved the following extension of an oscillatory result of Atkinson [**1**], who considered the special case $f(y) = y^{2n+1}$, $n = 1, 2, \ldots$.

THEOREM A. *Assume that for some $p > 1$, $f(y)$ satisfies*

$$\liminf_{y\to\infty} \frac{f(y)}{|y|^p} > 0.$$

*Then a necessary and sufficient condition that all solutions of* (3) *are oscillatory is that*

$$\int^{\infty} tq(t)\, dt = \infty. \tag{4}$$

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Removing the assumption $q(t) \geqslant 0$ from Waltman's theorem, Legatos and Kartsatos [7] proved the following result:

THEOREM B. *In addition to* (4) *assume that*

$$\int_1^{\infty} \frac{dt}{f(t)} < \infty, \qquad \int_{-1}^{-\infty} \frac{dt}{f(t)} < \infty. \tag{5}$$

*Then every solution of* (3) *is either oscillatory or tends monotonically to zero as* $t \to \infty$.

Under the same assumptions of Theorem B, Travis [**10**, Theorem 2.1] proved all solutions of (3) to be oscillatory.

Recently, the present author [**13**] gave a new criterion for the oscillation of (3) by removing the condition (5) and using the $n$th primitive of the coefficient $q(t)$ for some integer $n \geqslant 3$.

The purpose of this note is to establish some new oscillation criteria for the following more general nonlinear second order differential equation with damped term

$$y''(t) + p(t)y'(t) + q(t)f(y(t)) = 0 \tag{6}$$

where $p, q \in C[t_0, \infty), f \in C(R), yf(y) > 0$ for $y \neq 0$.

Results for (6) with nonlinear damping have been obtained by Baker [**2**], Bobisud [**3**], Butler [**4**] and Pintér [**9**].

By a solution of (6) at $t_0 \geqslant 0$ is meant a function $y$: $[t_0, t_1) \to R$, $t_0 < t_1$, which satisfies (6) for all $t \in [t_0, t_1)$. We assume the existence of solutions of (6) at $t_0$ for every $t_0 \geqslant 0$. A solution $y(t)$ of (6) at $t_0$ is said to be continuable if $y(t)$ exists for all $t \geqslant t_0$. A continuable solution $y(t)$ of (6) is called oscillatory if $y(t)$ has zeros for arbitrarily large $t$ and nonoscillatory if there exists $t^* \geqslant 0$ such that $y(t) \neq 0$ for all $t \geqslant t^*$.

**2. $q(t)$ is not assumed positive.** In this section, we treat the case that $q(t)$ is not assumed positive. At first, we give a new criterion for the oscillation of (6).

THEOREM 1. *Let $f'(y)$ exist and $f'(y) \geqslant k > 0$ for $y \in R' \equiv R - \{0\}$. If*

$$\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-u)^{n-1} u q(u)\, du = \infty, \tag{$C_1$}$$

$$\lim_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} u\left[(t-u)\left(p(u) - \frac{1}{u}\right) + n - 1\right]^2 (t-u)^{n-3}\, du < \infty \tag{$C_2$}$$

*for some integer $n \geqslant 3$, then every solution of* (6) *is oscillatory.*

PROOF. Let $y(t)$ be a nonoscillatory solution of (6) which, without loss of generality, we may assume $y(t) \neq 0$ for $t \geqslant t_0$. Define

$$w(t) = \frac{ty'(t)}{f(y(t))}.$$

Then $w(t)$ satisfies

$$w'(t) - \frac{w(t)}{t} + p(t)w(t) + q(t)t + w^2(t)\frac{f'(y(t))}{t} = 0.$$

This and $f'(y) \geqslant k > 0$ for $y \neq 0$ imply

$$w'(t) + \left[p(t) - \frac{1}{t}\right]w(t) + q(t)t + \frac{k}{t}w^2(t) \leqslant 0.$$

Thus

$$\int_{t_0}^{t}(t-u)^{n-1}w'(u)\,du + \int_{t_0}^{t}(t-u)^{n-1}\left(ku^{-1}w^2(u) + \left[p(u) - u^{-1}\right]w(u)\right)du$$
$$+\int_{t_0}^{t}(t-u)^{n-1}uq(u)\,du \leqslant 0.$$

Since

$$\int_{t_0}^{t}(t-u)^{n-1}w'(u)\,du = (n-1)\int_{t_0}^{t}(t-u)^{n-2}w(u)\,du - w(t_0)(t-t_0)^{n-1},$$

we get

$$\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-u)^{n-1}uq(u)\,du \leqslant w(t_0)\left(\frac{t-t_0}{t}\right)^{n-1}$$
$$-t^{1-n}\int_{t_0}^{t}\Big(ku^{-1}(t-u)^{n-1}w^2(u) + \big[(t-u)^{n-1}\left(p(u) - u^{-1}\right)$$
$$+(n-1)(t-u)^{n-2}\big]w(u)\Big)\,du$$
$$= w(t_0)\left(\frac{t-t_0}{t}\right)^{n-1} + \left(4kt^{n-1}\right)^{-1}$$
$$\cdot\int_{t_0}^{t}\left[(t-u)\left(p(u) - u^{-1}\right) + n - 1\right]^2 u(t-u)^{n-3}\,du$$
$$-t^{1-n}\int_{t_0}^{t}\Bigg[\left(ku^{-1}(t-u)^{n-1}\right)^{1/2}w(u)$$
$$+\frac{(t-u)\left(p(u) - u^{-1}\right) + n - 1}{2k^{1/2}}u^{1/2}(t-u)^{(n-3)/2}\Bigg]^2\,du$$
$$\leqslant w(t_0) + \left(4kt^{n-1}\right)^{-1}\int_{t_0}^{t}\left[(t-u)\left(p(u) - u^{-1}\right) + n - 1\right]^2 u(t-u)^{n-3}\,du$$
$$\to w(t_0) + M \equiv \text{a finite number},$$

as $t \to \infty$ by $(C_2)$, which contradicts condition $(C_1)$. This proves our theorem.

REMARK 1. It follows from $(C_2)$ that $p(t) \not\equiv 0$ in Theorem 1, in which $p(t)$ can be thought of as a small perturbation of $1/t$.

EXAMPLE 1. Consider the equation

$$\text{(E)} \qquad y''(t) + \frac{1}{t}y'(t) + \frac{1}{t^2}y(t) = 0, \qquad t \geqslant 1.$$

All conditions of Theorem 1 are satisfied for $n = 3$. Hence all solutions of equation (E) are oscillatory, whereas none of the known criteria [1], [7], [8], [10], [11] can obtain this result. One such solution of equation (E) is $y(t) = \sin(\ln t)$.

The following theorem extends the results of [1], [6], [12], [13] to equation (6) and consequently improves the results in [7], [8], [10].

THEOREM 2. *Let $f'(y)$ exist and $f'(y) \geqslant k > 0$ for $y \in R'$. If*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-u)^{n-1} q(u)\, du = \infty, \tag{C$_3$}$$

$$\lim_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} [(t-u)p(u) + n - 1]^2 (t-u)^{n-3}\, du < \infty \tag{C$_4$}$$

*for some integer $n \geqslant 3$, then every solution of* (6) *is oscillatory.*

PROOF. Let $y(t)$ be a nonoscillatory solution of (6), which without loss of generality, we may assume $y(t) \neq 0$ for $t \geqslant t_0$. Letting $w(t) = y'(t)/f(y(t))$, we have

$$w'(t) + w^2(t) f'(y(t)) + p(t)w(t) + q(t) = 0.$$

Thus

$$w'(t) + kw^2(t) + p(t)w(t) + q(t) \leqslant 0.$$

Hence

$$\int_{t_0}^{t} (t-u)^{n-1} w'(u)\, du + \int_{t_0}^{t} (t-u)^{n-1} \left[kw^2(u) + p(u)w(u)\right] du + \int_{t_0}^{t} (t-u)^{n-1} q(u)\, du \leqslant 0.$$

As in the proof of Theorem 1, we have

$$\begin{aligned} t^{1-n} \int_{t_0}^{t} (t-u)^{n-1} q(u)\, du &\leqslant w(t_0) \left( \frac{t-t_0}{t} \right)^{n-1} \\ &\quad - t^{1-n} \int_{t_0}^{t} \Bigg[ k^{1/2} (t-u)^{(n-1)/2} w(u) \\ &\qquad + \frac{(t-u)p(u) + n - 1}{2k^{1/2}} (t-u)^{(n-3)/2} \Bigg]^2 du \\ &\quad + (4kt^{n-1})^{-1} \int_{t_0}^{t} [(t-u)p(u) + n - 1]^2 (t-u)^{n-3}\, du \\ &\quad \to w(t_0) + M_0 \equiv \text{a finite number,} \end{aligned}$$

as $t \to \infty$, which contradicts condition ($C_3$). Thus our proof is complete.

REMARK 2. It follows from ($C_4$) that $p(t)$ may be equal to zero in Theorem 2, in which $p(t)$ can be thought of as a small perturbation of 0.

Taking $p(t) = 0$ in equation (6), we see easily that condition ($C_4$) can be removed and we have the following result:

COROLLARY 1 [13]. *Let $f'(y)$ exist and $f'(y) \geqslant k > 0$ for $y \in R'$. If* ($C_3$) *holds, then every solution of* (3) *is oscillatory.*

REMARK 3. Let $f(y) = y$ in Corollary 1. If (2) holds, then ($C_3$) holds for $n = 3$. Thus Wintner's result [12] is a special case of Corollary 1.

EXAMPLE 2. Consider the equation

$$\text{(F)} \qquad y''(t) + \frac{1}{2t}y'(t) + \frac{1}{4t}y(t) = 0, \qquad t \geqslant 1.$$

All conditions of Theorem 2 are satisfied for $n = 3$. Hence every solution of equation (F) is oscillatory, whereas none of the known criteria **[7]**, **[8]**, **[10]** can obtain this result. One such solution of equation (F) is $y(t) = 8\sin\sqrt{t}$.

**3. $q(t)$ is eventually nonnegative.** In this section, we discuss the case that $q(t)$ is eventually nonnegative and $f(y)$ is not required to be differentiable.

THEOREM 3. *Let $q(t) \geqslant 0$ and $f(y)/y \geqslant k > 0$ for $y \neq 0$. If* $(C_3)$ *and* $(C_4)$ *holds, then every solution of* (6) *is oscillatory.*

PROOF. Assume that $y(t)$ is a nonoscillatory solution of (6). Letting $w(t) = y'(t)/y(t)$, we have

$$w'(t) + w^2(t) + p(t)w(t) + q(t)\frac{f(y(t))}{y(t)} = 0.$$

Hence

$$w'(t) + w^2(t) + p(t)w(t) + kq(t) \leqslant 0.$$

Thus

$$\int_{t_0}^{t}(t-u)^{n-1}w'(u)\,du + \int_{t_0}^{t}(t-u)^{n-1}\left[w^2(u) + p(u)w(u)\right]du$$
$$+k\int_{t_0}^{t}(t-u)^{n-1}q(u)\,du \leqslant 0.$$

As in the proof of Theorem 1, we have

$$\frac{k}{t^{n-1}}\int_{t_0}^{t}(t-u)^{n-1}q(u)\,du \leqslant w(t_0)\left(\frac{t-t_0}{t}\right)^{n-1}$$
$$-t^{1-n}\int_{t_0}^{t}\left[(t-u)^{(n-1)/2}w(u) + \frac{(t-u)p(u)+n-1}{2}(t-u)^{(n-3)/2}\right]^2 du$$
$$+4^{-1}t^{1-n}\int_{t_0}^{t}[(t-u)p(u)+n-1]^2(t-u)^{n-3}\,du$$
$$\to w(t_0) + L \equiv \text{a finite number,}$$

as $t \to \infty$, which contradicts condition $(C_3)$. Thus our proof is complete.

COROLLARY 2. *Let $q(t) \geqslant 0$, $f(y)/y \geqslant k > 0$ for $y \neq 0$. If* $(C_3)$ *holds, then every solution of* (3) *is oscillatory.*

REMARK 4. The theorems and corollaries obtained in this note apply even when the weaker condition

$$\int_a^{\infty}\frac{dy}{f(y)} < \infty, \qquad \int_{-a}^{-\infty}\frac{dy}{f(y)} < \infty$$

fails for each $a > 0$; for example, $f(y) = y$ in equations (E) and (F).

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