OSCILLATION THEOREMS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DAMPED TERM

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Abstract. Some new integral criteria for the oscillation of the nonlinear second order differential equation with damped term \( y''(t) + p(t)y'(t) + q(t)f(y(t)) = 0 \) are given.

1. Introduction. Consider the linear differential equation

\[
y''(t) + a(t)y(t) = 0
\]

where \( a(t) \in C[t_0, \infty) \). By the well-known theorem of Wintner [12]

\[
\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} a(s) \, ds = \infty
\]

is sufficient for equation (1) to be oscillatory even when \( a(t) \) is not assumed positive. Hartman [5] proved that the limit cannot be replaced by the upper limit in condition (2). In [6], Kamenev extended Wintner's result by using the \( n \)th primitive

\[
A_n(t) = \frac{1}{n!} \int_{t_0}^{t} (t - u)^{n-1} a(u) \, du
\]

of the coefficient \( a(t) \) for some integer \( n \geq 3 \).

Let \( R \) be the set of all real numbers. Considering the nonlinear differential equation

\[
y''(t) + q(t)f(y(t)) = 0
\]

where \( q \in C[t_0, \infty), f \in C(R), yf(y) > 0 \) for \( y \neq 0 \) and \( f'(y) \geq 0 \) for all \( y \in R \). Under the assumption that \( q(t) \) is eventually nonnegative, Waltman [11] proved the following extension of an oscillatory result of Atkinson [1], who considered the special case \( f(y) = y^{2n+1}, n = 1, 2, \ldots \).

Theorem A. Assume that for some \( p > 1, f(y) \) satisfies

\[
\liminf_{y \to \infty} \frac{f(y)}{|y|^p} > 0.
\]

Then a necessary and sufficient condition that all solutions of (3) are oscillatory is that

\[
\int_{t_0}^\infty t q(t) \, dt = \infty.
\]
Removing the assumption \( q(t) \geq 0 \) from Waltman's theorem, Legatos and Kartsatos [7] proved the following result:

**Theorem B.** In addition to (4) assume that

\[
\int_{1}^{\infty} \frac{dt}{f(t)} < \infty, \quad \int_{-\infty}^{\infty} \frac{dt}{f(t)} < \infty.
\]

Then every solution of (3) is either oscillatory or tends monotonically to zero as \( t \to \infty \).

Under the same assumptions of Theorem B, Travis [10, Theorem 2.1] proved all solutions of (3) to be oscillatory.

Recently, the present author [13] gave a new criterion for the oscillation of (3) by removing the condition (5) and using the \( n \)th primitive of the coefficient \( q(t) \) for some integer \( n \geq 3 \).

The purpose of this note is to establish some new oscillation criteria for the following more general nonlinear second order differential equation with damped term

\[
y''(t) + p(t)y'(t) + q(t)f(y(t)) = 0
\]

where \( p, q \in C[t_{0}, \infty), f \in C(R), yf(y) > 0 \) for \( y \neq 0 \).

Results for (6) with nonlinear damping have been obtained by Baker [2], Bobisud [3], Butler [4] and Pinter [9].

By a solution of (6) at \( t_{0} > 0 \) is meant a function \( y: [t_{0}, t_{1}) \to R \), \( t_{0} < t_{1} \), which satisfies (6) for all \( t \in [t_{0}, t_{1}) \). We assume the existence of solutions of (6) at \( t_{0} \) for every \( t_{0} \geq 0 \). A solution \( y(t) \) of (6) at \( t_{0} \) is said to be continuable if \( y(t) \) exists for all \( t \geq t_{0} \). A continuable solution \( y(t) \) of (6) is called oscillatory if \( y(t) \) has zeros for arbitrarily large \( t \) and nonoscillatory if there exists \( t^{*} \geq 0 \) such that \( y(t) \neq 0 \) for all \( t \geq t^{*} \).

2. \( q(t) \) is not assumed positive. In this section, we treat the case that \( q(t) \) is not assumed positive. At first, we give a new criterion for the oscillation of (6).

**Theorem 1.** Let \( f'(y) \) exist and \( f'(y) \geq k > 0 \) for \( y \in R' \equiv R - \{0\} \). If

(C₁) \[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_{0}}^{t} (t-u)^{n-1} uq(u) \, du = \infty,
\]

(C₂) \[
\lim_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_{0}}^{t} u \left[ \left( p(u) - \frac{1}{u} \right) + n - 1 \right]^{2} (t-u)^{n-3} \, du < \infty
\]

for some integer \( n \geq 3 \), then every solution of (6) is oscillatory.

**Proof.** Let \( y(t) \) be a nonoscillatory solution of (6) which, without loss of generality, we may assume \( y(t) \neq 0 \) for \( t \geq t_{0} \). Define

\[
w(t) = \frac{ty'(t)}{f(y(t))}.
\]

Then \( w(t) \) satisfies

\[
w'(t) - \frac{w(t)}{t} + p(t)w(t) + q(t)t + w^{2}(t) \frac{f'(y(t))}{t} = 0.
\]
This and \( f'(y) \geq k > 0 \) for \( y \neq 0 \) imply
\[
w'(t) + \left[ p(t) - \frac{1}{t} \right] w(t) + q(t)t + \frac{k}{t} w^2(t) \leq 0.
\]

Thus
\[
\int_{t_0}^{t} (t-u)^{-1} w'(u) \, du + \int_{t_0}^{t} (t-u)^{-1} (ku^{-1}w^2(u) + \left[ p(u) - u^{-1} \right] w(u)) \, du
\]
\[
+ \int_{t_0}^{t} (t-u)^{-1} uq(u) \, du \leq 0.
\]

Since
\[
\int_{t_0}^{t} (t-u)^{-1} w'(u) \, du = (n-1) \int_{t_0}^{t} (t-u)^{-2} w(u) \, du - w(t_0)(t-t_0)^{n-1},
\]
we get
\[
\frac{1}{t^{n-1}} \int_{t_0}^{t} (t-u)^{-1} uq(u) \, du \leq w(t_0) \left( \frac{t-t_0}{t} \right)^{n-1}
\]
\[
- t^{1-n} \int_{t_0}^{t} (ku^{-1}(t-u)^{-1}w^2(u) + \left[ (t-u)^{-1} (p(u) - u^{-1})
\right.
\]
\[
+ (n-1)(t-u)^{n-2} w(u)) \, du
\]
\[
= w(t_0) \left( \frac{t-t_0}{t} \right)^{n-1} + (4kt^{n-1})^{-1}
\]
\[
\cdot \int_{t_0}^{t} [(t-u)(p(u) - u^{-1}) + n-1] \, du
\]
\[
- t^{1-n} \int_{t_0}^{t} \left[ (ku^{-1}(t-u)^{-1})^{1/2} w(u)
\right.
\]
\[
+ \frac{(t-u)(p(u) - u^{-1}) + n-1}{2k^{1/2}} \left[ u^{1/2}(t-u)^{(n-3)/2} \right]^{2} \, du
\]
\[
\leq w(t_0) + (4kt^{n-1})^{-1} \int_{t_0}^{t} [(t-u)(p(u) - u^{-1}) + n-1] \, du
\]
\[
\to w(t_0) + M \equiv \text{a finite number},
\]
as \( t \to \infty \) by \( \text{(C}_2\text{)} \), which contradicts condition \( \text{(C}_1\text{)} \). This proves our theorem.

**Remark 1.** It follows from \( \text{(C}_2\text{)} \) that \( p(t) \equiv 0 \) in Theorem 1, in which \( p(t) \) can be thought of as a small perturbation of \( 1/t \).

**Example 1.** Consider the equation
\[
y''(t) + \frac{1}{t} y'(t) + \frac{1}{t^2} y(t) = 0, \quad t \geq 1.
\]

All conditions of Theorem 1 are satisfied for \( n = 3 \). Hence all solutions of equation \( \text{(E)} \) are oscillatory, whereas none of the known criteria \[1\], \[7\], \[8\], \[10\], \[11\] can obtain this result. One such solution of equation \( \text{(E)} \) is \( y(t) = \sin(\ln t) \).
The following theorem extends the results of [1], [6], [12], [13] to equation (6) and consequently improves the results in [7], [8], [10].

**Theorem 2.** Let \( f'(y) \) exist and \( f'(y) \geq k > 0 \) for \( y \in \mathbb{R}' \). If

\[
(C_3) \quad \limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t-u)^{n-1} q(u) \, du = \infty, 
\]

\[
(C_4) \quad \lim_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^t [(t-u)p(u) + n-1]^2 (t-u)^{n-3} \, du < \infty 
\]

for some integer \( n \geq 3 \), then every solution of (6) is oscillatory.

**Proof.** Let \( y(t) \) be a nonoscillatory solution of (6), which without loss of generality, we may assume \( y(t) \neq 0 \) for \( t \geq t_0 \). Letting \( w(t) = y'(t)/f(y(t)) \), we have

\[
w'(t) + w^2(t)f'(y(t)) + p(t)w(t) + q(t) = 0.
\]

Thus

\[
w'(t) + kw^2(t) + p(t)w(t) + q(t) \leq 0.
\]

Hence

\[
\int_{t_0}^t (t-u)^{n-1} w'(u) \, du + \int_{t_0}^t (t-u)^{n-1} [kw^2(u) + p(u)w(u)] \, du \\
+ \int_{t_0}^t (t-u)^{n-1} q(u) \, du \leq 0.
\]

As in the proof of Theorem 1, we have

\[
t^{1-n} \int_{t_0}^t (t-u)^{n-1} q(u) \, du \leq w(t_0) \left( \frac{t-t_0}{t} \right)^{n-1} \\
- t^{1-n} \int_{t_0}^t \left[ k^{1/2}(t-u)^{(n-1)/2} w(u) \\
+ \frac{(t-u)p(u) + n-1}{2k^{1/2}} (t-u)^{(n-3)/2} \right]^2 \, du \\
+ (4kt^{n-1})^{-1} \int_{t_0}^t [(t-u)p(u) + n-1]^2 (t-u)^{n-3} \, du \\
\to w(t_0) + M_0 \equiv \text{a finite number},
\]

as \( t \to \infty \), which contradicts condition \((C_3)\). Thus our proof is complete.

**Remark 2.** It follows from \((C_4)\) that \( p(t) \) may be equal to zero in Theorem 2, in which \( p(t) \) can be thought of as a small perturbation of 0.

Taking \( p(t) = 0 \) in equation (6), we see easily that condition \((C_4)\) can be removed and we have the following result:

**Corollary 1** [13]. Let \( f'(y) \) exist and \( f'(y) \geq k > 0 \) for \( y \in \mathbb{R}' \). If \((C_3)\) holds, then every solution of (3) is oscillatory.

**Remark 3.** Let \( f(y) = y \) in Corollary 1. If (2) holds, then \((C_3)\) holds for \( n = 3 \). Thus Wintner’s result [12] is a special case of Corollary 1.
Example 2. Consider the equation

\[ (F) \quad y''(t) + \frac{1}{2t} y'(t) + \frac{1}{4t} y(t) = 0, \quad t \geq 1. \]

All conditions of Theorem 2 are satisfied for \( n = 3 \). Hence every solution of equation (F) is oscillatory, whereas none of the known criteria [7], [8], [10] can obtain this result. One such solution of equation (F) is \( y(t) = 8 \sin \sqrt{t} \).

3. \( q(t) \) is eventually nonnegative. In this section, we discuss the case that \( q(t) \) is eventually nonnegative and \( f(y) \) is not required to be differentiable.

Theorem 3. Let \( q(t) \geq 0 \) and \( f(y)/y \geq k > 0 \) for \( y \neq 0 \). If (C3) and (C4) holds, then every solution of (6) is oscillatory.

Proof. Assume that \( y(t) \) is a nonoscillatory solution of (6). Letting \( w(t) = y'(t)/y(t) \), we have

\[ w'(t) + w^2(t) + p(t)w(t) + q(t)f(y(t))/y(t) = 0. \]

Hence

\[ w'(t) + w^2(t) + p(t)w(t) + kq(t) \leq 0. \]

Thus

\[ \int_{t_0}^{t} (t-u)^{-1} w'(u) \, du + \int_{t_0}^{t} (t-u)^{-1} \left[ w^2(u) + p(u)w(u) \right] \, du \]

\[ + k \int_{t_0}^{t} (t-u)^{-1} q(u) \, du \leq 0. \]

As in the proof of Theorem 1, we have

\[ \frac{k}{t^{n-1}} \int_{t_0}^{t} (t-u)^{-1} q(u) \, du \leq w(t_0) \left( \frac{t-t_0}{t} \right)^{n-1} \]

\[ - t^{1-n} \int_{t_0}^{t} \left[ (t-u)^{n-1/2} w(u) + \frac{(t-u)p(u) + n-1}{2} (t-u)^{(n-3)/2} \right]^2 \, du \]

\[ + 4^{-1} t^{1-n} \int_{t_0}^{t} [(t-u)p(u) + n-1]^2 (t-u)^{n-3} \, du \]

\[ \rightarrow w(t_0) + L \equiv \text{a finite number}, \]

as \( t \to \infty \), which contradicts condition (C3). Thus our proof is complete.

Corollary 2. Let \( q(t) \geq 0 \), \( f(y)/y \geq k > 0 \) for \( y \neq 0 \). If (C3) holds, then every solution of (3) is oscillatory.

Remark 4. The theorems and corollaries obtained in this note apply even when the weaker condition

\[ \int_{a}^{\infty} \frac{dy}{f(y)} < \infty, \quad \int_{-\infty}^{a} \frac{dy}{f(y)} < \infty \]

fails for each \( a > 0 \); for example, \( f(y) = y \) in equations (E) and (F).
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REFERENCES

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