FIXED POINTS OF NONEXPANSIVE CONDENSING MULTI-VALUED MAPPINGS ON METRIC SPACES

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**Abstract.** In this paper we consider the existence of fixed points of nonexpansive condensing multi-valued mappings from a certain kind of metric space into itself; the spaces, here, are neither linear nor compact. Our result generalizes a theorem of Dotson and also a theorem of Bose and Mukherjee in some respect.

In this note we consider the existence of fixed points of nonexpansive condensing multi-valued mappings from a certain kind of metric space into itself. This result generalizes a theorem of Dotson [2] and also a theorem of Bose and Mukherjee [1] in some respect.

Let \((X, d)\) be a metric space. Denote by \(2^X\) the set of all closed subsets of \(X\) and by \(H\) the Hausdorff metric on \(2^X\) induced by metric \(d\).

**Definition 1.** If \(F = \{f_A | A \in 2^X\}\) is a family of functions from \([0, 1]\) into \(2^X\) satisfying the following conditions:

(I) \(f_A(1) = A\) for all \(A \in 2^X\),

(II) there is a function \(\phi: [0, 1) \rightarrow [0, 1)\) with

\[
H(f_A(t), f_B(t)) \leq \phi(t)H(A, B)
\]

for all \(A, B\) in \(2^X\) and \(t \in [0, 1)\),

(III) for all \(A \in 2^X\), all \(t_0 \in [0, 1]\),

\[
\lim_{t \to t_0} H(f_A(t), f_A(t_0)) = 0,
\]

and for all \(t \in [0, 1]\),

(IV) \(f_A(t) \subset f_B(t)\) for all \(A, B\) in \(2^X\) with \(A \subset B\), and

(V) \(f_A(t)\) is compact whenever the set \(A\) is finite;

we call \(F\) an \((S)\)-convex structure on \(X\).

**Remark 1.** If \(X\) is a starlike convex subset of a Banach space and if \(x_0\) is a starlike center of \(X\), then for each \(A \subset X\), we define

\[
f_A(t) = (1 - t)x_0 + tA, \quad t \in [0, 1].
\]

\(F = \{f_A | A \in 2^X\}\) is an \((S)\)-convex structure.
Let $\gamma$ denote the measure of noncompact sets on $X$ [4], that is, for each bounded subset $A$ of $X$,

$$\gamma(A) = \inf \{ r > 0 \mid A \text{ can be covered by finitely many balls of radii } \leq r \}.$$

**Definition 2.** Let $T: X \to 2^X$.
(1) $T$ is said to be nonexpansive if for any $x \in X$ and any $y \in X$, we have

$$H(Tx, Ty) \leq d(x, y).$$

(2) A continuous mapping $T$ of $(X, d)$ into $(2^X, H)$ is said to be condensing if for any bounded set $A \subset X$, we have

$$\gamma(T(A)) < \gamma(A)$$

provided that $\gamma(A) \neq 0$.

Our result is the following:

**Theorem.** Let $(X, d)$ be a bounded complete metric space with an $(S)$-convex structure $F$. Then every nonexpansive condensing multi-valued mapping $T$ of $X$ into $2^X$ has a fixed point.

Before we prove our theorem, we first need some preliminary results related to measure of noncompact sets as follows:

**Lemma 1.** Let $\gamma$ be defined as before. Then
(1) $\gamma(A) = 0$ if and only if $A$ is compact,
(2) $\gamma(\bigcup_{i=1}^{n} A_i) = \max\{\gamma(A_i) \mid 1 \leq i \leq n\}$,
(3) $\gamma(A) \leq \gamma(B)$ whenever $A \subset B$,
(4) $\gamma(A_r) \leq \gamma(A) + r$, where $r > 0$ and $(A)_r = \{b \in X \mid d(b, A) \leq r\}$,
(5) $\gamma(A) = \gamma(A)$ for all bounded subsets $A$ of $X$.

The proof of Lemma 1 is known (see Sadovskii [5]).

**Remark 2.** It is easy to see that $r > \gamma(A)$ if and only if there is a finite set $G$ such that $(G)_r \supset A$ if and only if there is a finite set $J$ such that $H(J, A) \leq r$.

In order to prove our theorem we need the following three lemmas.

**Lemma 2.** Suppose that $F$ satisfies (II) and (III) of Definition 1. Then for $\{A_n \mid n = 0, 1, 2, \ldots \} \subset 2^X$ and $\{t_n \mid n = 0, 1, 2, \ldots \} \subset [0, 1]$ with $\lim_{n \to \infty} H(A_n, A_0) = 0$ and $\lim_{n \to \infty} t_n = t_0$, we have

$$\lim_{n \to \infty} H(f_{A_n}(t_n), f_{A_0}(t_0)) = 0.$$

**Proof.** Since the triangle inequality is valid for the Hausdorff metric and (II) we have

$$H(f_{A_n}(t_n), f_{A_0}(t_0)) \leq H(f_{A_n}(t_n), f_{A_0}(t_n)) + H(f_{A_0}(t_n), f_{A_0}(t_0))$$

$$\leq H(A_n, A_0) + H(f_{A_0}(t_n), f_{A_0}(t_0));$$

let $n$ tend to $\infty$; it follows from (III) and the hypothesis that we have

$$\lim_{n \to \infty} H(f_{A_n}(t_n), f_{A_0}(t_0)) = 0.$$
In the rest of this paper we assume that $F$ is an $(S')$-convex structure on the metric space $(X, d)$.

**Lemma 3.** For $0 \leq t \leq 1$, the mapping $x \to f_{(x)}(t)$ is nonexpansive from $X$ into $2^X$. Moreover, for all $A \in 2^X$ and $t \in [0, 1]$

$$\gamma(A) \geq \gamma(f_A(t)).$$

**Proof.** Due to (II) it is easy to see that the mapping $x \to f_{(x)}(t)$ is nonexpansive.

Next, if $r > \gamma(A)$, by Remark 2 there is a finite set $G$ such that $H(G, A) \leq r$. By (V), $f_G(t)$ is compact, thus for any $c > 0$ there is a finite set $J$ with $H(J, f_G(t)) < c$.

By the triangle inequality and (II) we have

$$H(J, f_A(t)) \leq H(J, f_G(t)) + H(f_G(t), f_A(t)) < c + r.$$

By Remark 2, $\gamma(f_A(t)) < c + r$ for all $c > 0$ and $r > \gamma(A)$. Hence

$$\gamma(f_A(t)) < \gamma(A).$$

**Lemma 4.** For any $A \subseteq X$, let

$$g(A) = \bigcup \{ f_{(x)}(t) | 0 \leq t \leq 1 \}.$$

Then $\gamma(g(A)) = \gamma(A)$.

**Proof.** It is due to (5) in Lemma 1 and the definition of $g$, we may assume that $A \subseteq 2^X$. Define a real-valued function $h$ on $[0, 1] \times [0, 1]$ as follows:

$$h(s, t) = H(f_A(s), f_A(t)).$$

It follows from condition (III) that $h$ is a continuous function on $[0, 1] \times [0, 1]$, and thus $h$ is uniformly continuous. In particular, for $\epsilon > 0$ there is a positive number $\delta$ such that

$$h(s, t) < \epsilon \quad \text{for all } 0 \leq s, t \leq 1 \text{ and } |s - t| < \delta.$$  

Hence for all $0 \leq s, t \leq 1$ and $|s - t| < \delta$ we have $f_A(s) \subseteq (f_A(t))_\epsilon$. Take $P = \{ t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1 \}$ be a partition of $[0, 1]$ with

$$t_i - t_{i-1} < \delta \quad \text{for all } i = 1, 2, \ldots, n,$$

then $g(A) \subseteq \bigcup_{i=1}^n (f_A(t_i))_\epsilon$.

From Lemmas 1 and 3, we have

$$\gamma(g(A)) \leq \gamma \left( \bigcup_{i=1}^n (f_A(t_i))_\epsilon \right) = \gamma \left( \left( f_A(t_i) \right)_\epsilon \right) \leq \gamma(A) + \epsilon.$$  

Therefore, $\gamma(g(A)) < \gamma(A)$. The fact that $A \subseteq g(A)$ implies

$$\gamma(g(A)) = \gamma(A).$$

**Remark 3.** It follows from (IV) and the definition of $g$ that for $A \subseteq B \subseteq X$ we have

$$f_A(t) \subseteq f_B(t) \quad \text{for all } t \in [0, 1].$$
and then
\[ g(A) \subseteq g(B). \]

Now we are going to prove our theorem.

**Proof of the theorem.** Let \( x_0 \in X. \) We denote \( O_{x_0} \) as the set
\[ \bigcup \{ (gT)^n x_0 \mid n = 0, 1, 2, \ldots \} \]
where \( (gT)^0 x_0 = \{x_0\} \) and \( (gT)^n x_0 = g(T((gT)^{n-1} x_0)) \) for \( n = 1, 2, \ldots \). Since \( T(gT)^n x_0 \subseteq (gT)^{n+1} x_0 \), we have
\[ T(\overline{O}_{x_0}) \subseteq \overline{O}_{x_0} \quad \text{and} \quad \gamma(T(\overline{O}_{x_0})) \leq \gamma(\overline{O}_{x_0}). \]
Moreover, by (IV) and Remark 3 that
\[ g(T(\overline{O}_{x_0})) = g\left( \bigcup_{n=0}^{\infty} T(gT)^n x_0 \right) \subseteq (gT)^{k+1} x_0 \]
for all \( k = 0, 1, 2, \ldots \); hence
\[ \{x_0\} \cup gT(O_{x_0}) \supseteq \{x_0\} \cup \left( \bigcup_{n=0}^{\infty} (gT)^{n+1} x_0 \right) = O_{x_0} \]
and thus
\[ \gamma(\overline{O}_{x_0}) = \gamma(O_{x_0}) \leq \gamma(gT(\overline{O}_{x_0})) = \gamma(T(\overline{O}_{x_0})). \]
That is,
\[ \gamma(\overline{O}_{x_0}) = \gamma(T(\overline{O}_{x_0})). \]
Since \( T \) is a condensing mapping, we have that \( \overline{O}_{x_0} \) is a compact subset of \( X \). Now we let \( \{r_n\} \) be a strictly increasing sequence of positive numbers with \( \lim_{n \to \infty} r_n = 1 \) and let \( K = \overline{O}_{x_0} \) and
\[ T_n x = f_{T_{r_n}}(r_n) \quad \text{for all } x \in K. \]
For \( y \in O_{x_0} \) we have \( y \in (gT)^k x_0 \) for some \( k = 0, 1, 2, \ldots \). It follows from the definitions of \( T_n \) and \( O_{x_0} \) and Remark 3 that
\[ T_n y = f_{T_y}(r_n) \subseteq g(T_y) \subseteq (gT)^{k+1} x_0 \subseteq O_{x_0}. \]
Then for any \( x \in K \) there is a sequence \( \{y_i\} \) in \( O_{x_0} \) which converges to \( x \). By (II) we have
\[ H(T_n y_i, T_n x) = H(f_{T_{y_i}}(r_n), f_{T_x}(r_n)) \leq H(Ty_i, Tx) \leq d(y_i, x); \]
hence \( \lim_{i \to \infty} H(T_n y_i, T_n x) = 0 \), and \( T_n x \in \overline{O}_{x_0} = K \), and thus \( T_n \) maps \( K \) into \( 2^K \). Moreover, if \( x, y \) are in \( K \) we have
\[ H(T_n x, T_n y) = H(f_{T_x}(r_n), f_{T_y}(r_n)) \leq \phi(r_n) d(x, y); \]
that is, \( T_n \) is a contraction mapping of \( K \) into \( 2^K \). It is due to a theorem of Nadler [3] that \( T_n \) has a fixed point \( x_n \) in \( K \), or \( x_n \in f_{T_{r_n}}(r_n) \). By the compactness of \( K \) there is a
convergent subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \), with \( y_0 \) as its limit. Hence, by Lemma 2, we have that
\[
\lim_{k \to \infty} H(f_{T_{x_{n_k}}} (r_{n_k}), f_{T_{y_0}} (1)) = 0
\]
and therefore \( y_0 \in f_{T_{y_0}} (1) = T y_0 \).

It follows from our theorem and Remark 1 that we have the following:

**Corollary.** Let \( X \) be a starlike convex, bounded, closed subset of a Banach space \( E \). Then every nonexpansive condensing multi-valued mapping of \( X \) into \( 2^X \) has a fixed point.

**Remark 4.** If \( X \) is compact then every continuous mapping of \( X \) into \( 2^X \) is condensing, hence Dotson's theorem and its Corollary 2 are special cases of our theorem and corollary, respectively.

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**References**


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