

THE COTYPESET OF A TORSION FREE ABELIAN GROUP OF RANK TWO

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ABSTRACT. The cotyposet (set of types of rank one factors) of a torsion free abelian group of rank two is characterized.

Let G be a torsion free abelian group of finite rank, hereafter called simply a "group". The *typeset* of G , $\{\text{type}(x) \mid 0 \neq x \in G\}$, has been studied by several authors (e.g. [3, 5, 7, 8, 9, 10]). However, the main problem: When is a set of types the typeset of a group G ?, has not yet been solved, even for groups of rank two. The *cotyposet* of G , introduced in [10], is the set of types of all rank one factors of G . This set also seems to be important in the study of torsion free groups (see e.g. [1, 2, 11, 12, 13]). It therefore seems of interest to characterize those sets of types which are the cotyposet of a group G . In this paper we give necessary and sufficient conditions for a set of types to be the cotyposet of a rank two group.

Familiarity is assumed with the notions of type and characteristic (height vector) —see [6]. In particular, the inner type, IT, and outer type OT, of Warfield are used [13]. The symbols \vee and \wedge are used to denote the sup and inf, respectively, of collections of characteristics. If t is a characteristic, t^p denotes the value of t at the prime p . If τ is a type, τ^p is called finite (infinite), if t^p is finite (infinite) for some characteristic $t \in \tau$. If τ_1 and τ_2 are types with $\tau_1 \leq \tau_2$, then $\tau_2 - \tau_1$ is the type of $t_2 - t_1$, where $t_1 \in \tau_1$, $t_2 \in \tau_2$ are characteristics chosen so that $t_1 \leq t_2$ (note: $\infty - \infty = 0$). The type of Z is denoted by $\mathbf{0}$. Finally, we call two types, τ_1 and τ_2 , equivalent on a subset P of the primes ($\tau_1 \sim \tau_2$ on P) if there exist characteristics $t_1 \in \tau_1$, $t_2 \in \tau_2$ such that $t_1^p = t_2^p$ for all $p \in P$.

The characterization of rank two groups by Beaumont-Wisner [4] is employed repeatedly. As usual, $h_G^p(x)$ denotes the p -height of an element x in a group G , and $\text{type}_G(x)$ is the type of x in G . We use $\langle \rangle$ ($\langle \rangle_*$) to denote the subgroup (pure subgroup) generated by a set of elements.

We begin with a simple lemma.

LEMMA 1. Let $S = \{\sigma_i\}_{i \in I}$ be a set of types and σ_0 a type such that $\sigma_0 = \sigma_i \vee \sigma_j$ for all $i \neq j$ in I . Then

- (a) $\sigma_0 - \sigma_i \leq \sigma_j$ for all $i \neq j$ in I .
- (b) $(\sigma_0 - \sigma_i) \wedge (\sigma_0 - \sigma_j) = \mathbf{0}$ for all $i \neq j$ in I .

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PROOF. Obvious.

A special case of the main theorem is proved first.

THEOREM 1. *Let $S = \{\sigma_i\}$ be a finite or countable set of types. Then S is the cotypeset of a group G of rank two with $IT(G) = \mathbf{0}$ if and only if there exists a type σ_0 such that:*

- (1) $\sigma_0 = \sigma_i \vee \sigma_j$ for all $i \neq j$, and if S is finite, $\sigma_0 = \sigma_i$ for some i .
- (2) $\sigma_j - (\sigma_0 - \sigma_i) = \sigma_i - (\sigma_0 - \sigma_j)$ for all $i \neq j$.

PROOF. I. Necessity. Suppose S is the cotypeset of a rank two group G with $IT(G) = \mathbf{0}$. Let $\sigma_0 = OT(G)$, the outer type of G . Then the first claim of condition (1) is immediate. For all i , choose $x_i \in G$ such that $G/\langle x_i \rangle_*$ has type σ_i . If $S = \{\sigma_1, \dots, \sigma_n\}$ is finite, choose $0 \neq x \in G$ such that $\langle x \rangle_* \neq \langle x_i \rangle_*$, $i = 1, \dots, n$. Let $\sigma = \text{type } G/\langle x \rangle_*$. Then $\sigma \in S$ and $\sigma \vee \sigma_i = \sigma_0$ for all i . Thus, $\sigma = \sigma_0$.

To establish the remaining claim, for each i define $\tau_i = \sigma_0 - \sigma_i$ and let $\tau'_i = \text{type } x_i$. It is well known that since G is rank two of inner type $\mathbf{0}$, $\tau'_i + \sigma_i = \sigma_0$. It follows that we can choose characteristics $t_i \in \tau_i$, $t'_i \in \tau'_i$ with $t_i^p = (t'_i)^p$ for all primes p except possibly those for which $\sigma_i^p = \sigma_0^p = \infty$. At all such p we can set $t_i^p = 0$, although $(t'_i)^p$ can be any nonnegative integer. Thus $t_i \leq t'_i$, but equality need not hold.

If $i \neq j$, let $X_i = G/\langle x_i \rangle_*$, $X_j = G/\langle x_j \rangle_*$. Then X_i and X_j are rank one groups such that $X_i \supseteq \langle x_j \rangle_*$, $X_j \supseteq \langle x_i \rangle_*$ under the natural embeddings. It is well known (see e.g. [4]) that $X_j/\langle x_i \rangle_* \cong X_i/\langle x_j \rangle_*$. The isomorphism implies that we can choose characteristics $s_i \in \sigma_i$, $s_j \in \sigma_j$ with $s_j - t'_i = s_i - t'_j$. Thus, in view of our earlier remarks, if p is a prime with s_i^p and s_j^p finite, then $t_i^p = (t'_i)^p$, $t_j^p = (t'_j)^p$, and $s_j^p - t_i^p = s_i^p - t_j^p$. If $s_i^p = s_j^p = \infty$, then $t_i^p = t_j^p = 0$, so again $s_j^p - t_i^p = s_i^p - t_j^p$. Let p be a prime for which $s_i^p = \infty$, $s_j^p < \infty$. Then $\infty = s_0^p = s_j^p + (t'_j)^p$, so $(t'_j)^p = \infty$. Thus, $0 = s_i^p - (t'_j)^p = s_j^p - (t'_i)^p$. Moreover, $\tau'_i \wedge \tau'_j = \mathbf{0}$, so $\tau'_i \sim \mathbf{0}$ on the set of all primes p where $(\tau'_j)^p = \infty$.

Since $\tau_i \leq \tau'_i$ the same claim holds for τ_i . Thus, the equation $s_j^p - t_i^p = s_i^p - t_j^p$ holds for almost all p where $s_i^p = \infty$, $s_j^p < \infty$, and where the equality fails both sides are finite. By symmetry the same statement is true for the set of primes where $s_i^p < \infty$, $s_j^p = \infty$. This establishes condition (2).

II. Sufficiency.

A. Choosing characteristics. Let $S = \{\sigma_i\}$ be a finite or countable set of types satisfying conditions (1) and (2). Define $\tau_i = \sigma_0 - \sigma_i$ as before. We show $\{\tau_i\}$ is relatively disjoint, that is, there exists a set of characteristics $t_i \in \tau_i$ such that $t_i^p \wedge t_j^p = 0$ for all primes p and $i \neq j$. Suppose $t_1 \in \tau_1, \dots, t_n \in \tau_n$ have been chosen such that:

- (i) for all p and $1 \leq i < j \leq n$, $t_i^p \wedge t_j^p = 0$,
- (ii) $t_i^p = 0$ if $\tau_k^p = \infty$ for any $k \in \mathbb{Z}^+$, $k \neq i$.

Condition (ii) is possible since if $\tau_k^p = \infty$ then σ_0^p and σ_i^p must both be infinite by Lemma 1(a) and Condition (1) of the theorem. Thus $\tau_i = \sigma_0 - \sigma_i$ has a characteristic which is 0 at all such p . Choose $t'_{n+1} \in \tau_{n+1}$. Note that if $(t'_{n+1})^p = \infty$ for some p then $t_i^p = 0$ for $1 \leq i \leq n$. Furthermore, $\Pi_i = \{p \mid 0 < (t'_{n+1})^p \wedge t_i^p < \infty\}$ is finite

for each i by Lemma 1(b). Define $t_{n+1}^p = 0$ if $p \in \cup_{i=1}^n \Pi_i$ or if $\tau_k^p = \infty$ for some $k \in Z^+, k \neq n + 1$; and $t_{n+1}^p = (t_{n+1}^i)^p$ otherwise. Then $t_{n+1} \in \tau_{n+1}$ and t_1, \dots, t_{n+1} satisfy (i) and (ii). By induction, a set $\{t_i\}$ of characteristics can be chosen with $t_i \in \tau_i$ and $t_i^p \wedge t_j^p = 0$ for all $p, i \neq j$.

B. Ito's construction [7]. Given a relatively disjoint set of types $\{\tau_i\}$, Ito constructs a rank two group G with typeset $G = \{\tau_i\}$ as follows. Let $t_i \in \tau_i$ be a set of relatively disjoint characteristics. For each i , choose a rank one group G_i of type τ_i , and $x_i \in G_i$ whose characteristic is exactly t_i . Let $\{(a_i, b_i) \mid 1 \leq i < \infty\}$ be the set of all coprime pairs of integers with $b_i \geq 0$, numbered so that $\text{Max}\{|a_i|, b_i\} \leq i$. Then if $H = \oplus_{i=3}^\infty \langle a_i x_1 + b_i x_2 - x_i \rangle$ and $G = \oplus_{i=1}^\infty G_i / H$, Ito shows that G is a torsion free rank two group, and that the element $g_i = a_i \bar{x}_1 + b_i \bar{x}_2 = a_i(x_1 + H) + b_i(x_2 + H)$ has type τ_i in G .

Two additional facts about Ito's construction can be noted. First, he considers only the case when $\{\tau_i\}$ is infinite, the finite case having been settled in [3]. However, his construction also works if $\{\tau_i\}$ is finite. In this case, all but a finite number of the elements $\{g_i\}$ are assigned the type $\mathbf{0}$, which is necessarily in element of $\{\tau_i\}$. Second, an easy computation shows that the outer type of the rank two group constructed, $\text{OT}(G)$, is the type represented by the characteristic $\vee_i t_i$.

Henceforth we assume $S = \{\sigma_i\}$ is a set of types satisfying conditions (1) and (2) of the theorem, $\tau_i = \sigma_0 - \sigma_i$ and $\{t_i\}$ is a relatively disjoint set of characteristics with $t_i \in \tau_i$. Let G_1 be the group constructed by Ito's method with typeset $G_1 = \{\tau_i\}$, and let $\sigma = \text{OT}(G_1)$ with representative $s = \vee_i t_i$.

C. Changing $\text{OT}(G_1)$. In general $\sigma \leq \sigma_0$, since each $\tau_i \leq \sigma_0$. Choose $s_0 \in \sigma_0$ with $s^p \leq s_0^p$ for all primes p . Let $P_1 = \{p \mid 0 < s_0^p - s^p < \infty\}$, $P_2 = \{p \mid s^p < \infty, s_0^p = \infty\}$. In this section we construct a rank two group $G \geq G_1$ such that:

- (i) $\text{IT}(G) = \mathbf{0}$,
- (ii) $\text{OT}(G) = \sigma_0$,
- (iii) for all g_i , $\text{type}_G(g_i) \sim \text{type}_{G_1}(g_i)$ on the complement of P_2 (denoted P_2^c),
- (iv) for all g_i and $p \in P_2$, $h_G^p(g_i) < \infty$.

First note that if $|P_1| < \infty$ we already have $\sigma \sim \sigma_0$ on P_2^c . In this case we leave the p height of all g_i unchanged for $p \in P_2^c$. Otherwise, index the primes in P_1 in their natural order, say $P_1 = \{p_j\}$. It is easy to see that we can select a subsequence (a_j, b_j) , with all terms distinct, from our original sequence (a_i, b_i) of coprime pairs of integers, such that: $\lim_{j \rightarrow \infty} j/p_j = 0$ and if $t_i^{p_j} > 0$ then

$$\det \begin{pmatrix} a_j & b_j \\ a_i & b_i \end{pmatrix} \equiv 0 \pmod{p_j}.$$

(Note that for each $p_j \in P_1$ there is at most one t_i with $t_i^{p_j} > 0$.) Let

$$G_2 = \langle G_1, \{g_j/p_j^{n_j} \mid p_j \in P_1\} \rangle,$$

where $n_j = s_0^{p_j} - h_{G_1}^{p_j}(g_j)$. It is apparent that, for $p \notin P_1$, the p height of any g_i is unchanged in going from G_1 to G_2 . Furthermore, if $h_{G_2}^{p_j}(g_i) > 0$ then

$$\det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \equiv 0 \pmod{p_j}$$

[5, part I]. But $|a_i b_j - a_j b_i| \leq 2ij$ and $\lim_{j \rightarrow \infty} j/p_j = 0$. Thus, we can have $h_{G_2}^p(g_i) > 0$ for only finitely many $p_j \in P_1$, and $\text{type}_{G_2}(g_i) = \text{type}_{G_1}(g_i)$ for all g_i . Since each element on G_2 is rationally dependent on some g_i , it follows that typeset $G_2 =$ typeset G_1 . Finally, it is easy to see that the outer type of G_1 has been increased so that $\text{OT}(G_2) \sim \sigma_0$ on P_2^c .

We now consider the set of primes P_2 , on which the outer type must be made infinite. Let $X_1 = G_2/\langle \bar{x}_1 \rangle_*$ and $X_2 = G_2/\langle \bar{x}_2 \rangle_*$. As in I (Necessity), $X_1/\langle \bar{x}_2 \rangle_* \cong_{\theta} X_2/\langle \bar{x}_1 \rangle_*$, where θ is uniquely determined by G_2 . Let R be the subring of Q generated by $\{1/p \mid p \in P_2\}$. By definition of P_2 , none of the rank one groups $X_1, \langle \bar{x}_1 \rangle_*, X_2, \langle \bar{x}_2 \rangle_*$ is p -divisible for $p \in P_2$. Thus $R \otimes_Z X_1/\langle \bar{x}_2 \rangle_* \cong_{\theta'} R \otimes_Z X_2/\langle \bar{x}_1 \rangle_*$ and the isomorphism θ' can be chosen so that

- (*) θ' is an irrational p -adic unit on each p component such that $p \in P_2$.
- (**) The natural diagram commutes:

$$\begin{array}{ccc} X_1/\langle \bar{x}_2 \rangle_* & \cong_{\theta} & X_2/\langle \bar{x}_1 \rangle_* \\ \downarrow & & \downarrow \\ R \otimes_Z X_1/\langle \bar{x}_2 \rangle_* & \cong_{\theta'} & R \otimes_Z X_2/\langle \bar{x}_1 \rangle_* \end{array}$$

Let

$$G = \left\{ (\alpha, \beta) \mid \alpha \in R \otimes X_1, \beta \in R \otimes X_2 \text{ and } \theta'(\alpha + \langle \bar{x}_2 \rangle_*) = \beta + \langle \bar{x}_1 \rangle_* \right\}.$$

The following facts about G are easily verified:

- (a) The map $\beta \bar{x}_1 + \alpha \bar{x}_2 \rightarrow (\alpha, \beta)$ gives a natural embedding of G_2 into G (use (**)).
- (b) Under this embedding $h_{G_2}^p(x) = h_G^p(x)$ for all $x \in G_2$ and $p \notin P_2$.
- (c) $h_G^p(x) < \infty$ for all $0 \neq x \in G$ and $p \in P_2$ (use (*)).
- (d) $\text{OT}(G)$ is infinite at all primes $p \in P_2$.
- (e) $\text{OT}(G) \sim \text{OT}(G_2)$ on P_2^c .
- (f) $\text{IT}(G) = \mathbf{0}$. (The embedding maps $\langle \bar{x}_1 \rangle_*$ and $\langle \bar{x}_2 \rangle_*$ onto distinct pure subgroups of G .)

In view of (a)–(f), G is a group satisfying conditions (i)–(iv) stated at the beginning of this section.

D. The cotypeset of G . In this section we complete the proof of Theorem 1 by showing that the cotypeset of G is exactly S . By the results of the previous section, $\text{OT}(G) = \sigma_0$ and the typeset of G is $\{\tau_i'\}$ where $\tau_i' \sim \tau_i = \sigma_0 - \sigma_i$ on P_2^c and $\tau_i' < \infty$ at all primes $p \in P_2$. For each i , let g_i be an element of G with type τ_i' . Then, $\text{type } G/\langle g_i \rangle_* = \sigma_i'$ where $\sigma_i' + \tau_i' = \sigma_0$. Moreover, for all $i \neq j$, $\sigma_i' - \tau_j' = \sigma_j' - \tau_i'$ and $\tau_i' \wedge \tau_j' = \mathbf{0}$. We will show $\sigma_i' = \sigma_i$ for all i . The proof is divided into four cases.

Case I. $p \in P_2$. For $p \in P_2$, σ_0^p is infinite while τ_i^p and $\tau_i'^p$ are both finite. Thus both σ_i^p and $\sigma_i'^p$ are infinite on P_2 .

Case II. σ_0^p is finite. Let $P_3 = \{p \mid \sigma_0^p < \infty\}$. Then $\tau_i \sim \tau_i'$ on P_3 since $P_3 \subseteq P_2^c$. Since $\sigma_0 = \sigma_i + \tau_i = \sigma_i' + \tau_i'$, then $\sigma_i \sim \sigma_i'$ on P_3 .

Case III. $\sigma_0^p = \infty, p \notin P_2, \sigma_i^p = \infty$. At these primes, $\tau_i' \sim \tau_i \sim \mathbf{0}$, and hence σ_i' is infinite and equal to σ_i .

Case IV. $\sigma_0^p = \infty, \sigma_i^p < \infty, p \notin P_2$.

For each fixed i , this case can occur for only a finite number of primes p . This follows from the equations, $\sigma_i - \tau_j = \sigma_j - \tau_i$ (Condition (2)) and $\tau_i \wedge \tau_j = \mathbf{0}$ (Lemma 1(b)). Since τ_i infinite implies τ_i' infinite, then σ_i' is finite and hence equivalent to σ_i on this finite set of primes. Cases I–IV imply $\sigma_i = \sigma_i'$ for all i .

REMARK 1. In the proof of sufficiency (II) we have tacitly assumed $S = \{\sigma_i\}$ contains more than one element. If $S = \{\sigma_0\}$ is a set containing only one type, it is easy to construct a rank two group G , homogeneous of type $\mathbf{0}$, whose cotypeset is S .

REMARK 2. Theorem 1 would be a trivial consequence of §§5 and 6 of [10] if the results therein were true.

However, as the following example shows, the results in [10] are incorrect.

EXAMPLE. The following adaptation of an example of Dubois shows that conditions (1)–(4) of Proposition 4 of [10] are not sufficient to guarantee that a set of type-cotype pairs $\{(\tau_i, \sigma_i) \mid 1 \leq i < \infty\}$ is the type-cotype set of a rank two group. Let P be the set of primes in Z and write $P = \bigcup_{i=1}^{\infty} P_i$ as a disjoint union of countable sets P_i . Let t_1 be the characteristic defined by $t_1^p = 1$ for $p \in P_1, t_1^p = 0$ otherwise. For $i > 1$ define $t_i^p = 1$ for $p \in P_i, t_i^p = \infty$ if p is the i th prime in P_j for some $j < i, t_i^p = 0$ otherwise. Let τ_i be the type of t_i for $1 \leq i < \infty$, and let $\sigma_0 = \text{type } Q, \sigma_i = \sigma_0 - \tau_i$ for $1 \leq i < \infty$. It is easy to check that the set $\{(\tau_i, \sigma_i) \mid 1 \leq i < \infty\}$ satisfies (1)–(4) of Proposition 4 of [10]. However, as shown by Dubois, $\{\tau_i \mid 1 \leq i < \infty\}$ is not the type set of any rank two group.

A final lemma precedes the Main Theorem.

LEMMA 2. Let $S = \{\sigma_i\}$ be a finite or countable set of types. Then S is the cotypeset of a rank two group G if and only if there exists a type τ_0 such that (1) $\tau_0 \leq \sigma_i$ for all i , (2) $\{\sigma_i - \tau_0\}$ is the cotypeset of a rank two group of inner type $\mathbf{0}$. (That is, $\{\sigma_i - \tau_0\}$ satisfies the conditions of Theorem 1.)

PROOF. Suppose $S = \{\sigma_i\}$ is the cotypeset of a rank two group G . Let $\tau_0 = \text{IT}(G)$. Clearly, $\tau_0 \leq \sigma_i$ for all i . As in [10], G can be written $G = U \otimes G_0$, where U is a rank one group of type τ_0 and G_0 is a rank two group of inner type $\mathbf{0}$. Furthermore, G_0 can be chosen so that $\text{OT}(G_0) \sim \mathbf{0}$ on the set of primes p where τ_0^p is infinite. Let K be a pure rank one subgroup of G . Then there is an obvious isomorphism: $G/K \cong U \otimes G_0/U \otimes K_0 \cong U \otimes (G_0/K_0)$, where K_0 is a pure rank one subgroup of G_0 . Conversely, if K_0 is a pure rank one subgroup of G_0 , then $K = U \otimes K_0$ is a pure rank one subgroup of $G = U \otimes G_0$, and the same isomorphism holds. Furthermore, $\text{type } G/K = \tau_0 + \text{type } G_0/K_0$, and therefore, $\text{type } G_0/K_0 = \text{type } G/K - \tau_0$, in view of our choice of G_0 . It follows that cotypeset G_0 is exactly $\{\sigma_i - \tau_0\}$.

Conversely, suppose there exists a type τ_0 such that $\{\sigma_i - \tau_0\}$ is the cotypeset of a rank two group G_0 of inner type $\mathbf{0}$. If $\tau_0 < \text{type } Q$ let U be a rank one group of type τ_0 . Then $G = U \otimes G_0$ is a (reduced) group G with cotypeset $\{\sigma_i\}$. If $\tau_0 = \text{type } Q$, then $S = \{\text{type } Q\}$ by (1) and (2) holds trivially. As mentioned previously, in this case it is not difficult to construct the desired rank two group G .

THEOREM 2. *Let $S = \{\sigma_i\}$ be a finite or countable set of types. Then S is the cotypeset of a rank two group G if and only if there exists a type σ_0 such that $\sigma_i \vee \sigma_j = \sigma_0$ if $i \neq j$, subject to the additional requirement that $\sigma_0 \in S$ if $|S| < \infty$.*

PROOF. The condition is clearly necessary. To prove sufficiency, we produce a type τ_0 such that $\{\sigma_i - \tau_0\}$ satisfies the conditions of Theorem 1 and apply Lemma 2.

In view of our earlier remarks, we may assume $|S| > 1$. Choose $s_0 \in \sigma_0$, $s_1 \in \sigma_1$ with $s_1^p \leq s_0^p$ for all p . If $\{s_1, \dots, s_n\}$ have been chosen, choose $s_{n+1} \in \sigma_{n+1}$ with $s_{n+1}^p \leq s_0^p$, for all p , and with $s_{n+1}^p \vee s_j^p = s_0^p$, for all p and $1 \leq j \leq n$. This choice is possible since $\sigma_{n+1} \vee \sigma_j = \sigma_0$, $1 \leq j \leq n$. Thus, by induction, we have a set of characteristics $s_i \in \sigma_i$ with $s_i \vee s_j = s_0$. Note that for all p there is at most one s_i with $s_i^p < s_0^p$. Let t_0 be the characteristic defined by $t_0^p = \bigwedge s_i^p$ and let τ_0 be the type represented by t_0 . It is straightforward to verify that $\{\sigma_i' = \sigma_i - \tau_0 \mid 1 \leq i < \infty\}$ satisfies conditions (1) and (2) of Theorem 1 with respect to the type $\sigma_0' = \sigma_0 - \tau_0$. The Theorem now follows from Lemma 2.

The inner type τ_0 is not, in general, uniquely determined by the cotypeset S , due to the different possible choices for the characteristics $s_i \in \sigma_i$. A characterization of the possible inner types for a group G with cotypeset S is given by

THEOREM 3. *Let $S = \{\sigma_i\}$ and τ_0 be as in Theorem 2. Let $S_i = \{p \mid s_i^p = t_0^p < \infty \text{ and } s_0^p = \infty\}$. Then a type τ with characteristic t is the inner type of a rank 2 group G with cotypeset S if and only if for all i , $\tau \leq \sigma_i$ and $\{p \mid t^p \neq t_0^p\} \cap S_i$ is finite.*

PROOF. If τ is a type satisfying these hypotheses, it is easy to show that $\{\sigma_i - \tau\}$ satisfies the conditions of Theorem 1. Thus, as in the proof of Lemma 2, there is a rank 2 group G of inner type τ and cotypeset S .

Conversely, if τ is the inner type of a rank 2 group with cotypeset S , then $\{\sigma_i - \tau\}$ satisfies the conditions of Theorem 1.

In particular, if $i \neq j$, then

$$(\dagger) \quad (\sigma_j - \tau) - [(\sigma_0 - \tau) - (\sigma_i - \tau)] = (\sigma_i - \tau) - [(\sigma_0 - \tau) - (\sigma_j - \tau)].$$

Let $P_i = S_i \cap \{p \mid t^p \neq t_0^p\}$. Then, on this set of primes the left hand side of (\dagger) is $\sim \mathbf{0}$, while the right-hand side is $\sim (\sigma_i - \tau)$. Since, on P_i , $\sigma_i \sim \tau_0$, we have $\tau_0 - \tau \sim \mathbf{0}$ on this set. It follows that P_i must be finite.

REMARK 3. The “dual” of Theorem 2 is false. There exists a set of types $T = \{\tau_i \mid 1 \leq i < \infty\}$ and a type τ_0 such that $\tau_0 = \tau_i \wedge \tau_j$ for all $i \neq j$, but T is not the typeset of a rank two group [5, Example 1].

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