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ON THE FIRST FACTOR OF THE CLASS NUMBER OF A CYCLOTOMIC FIELD

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ABSTRACT. Let p be an odd prime. $h_1(p)$ is the first factor of the class number of field $Q(\zeta_p)$. We proved that

$$h_1(p) \leqslant \begin{cases} 2p\left(\dfrac{p-1}{8(2^{l/2}+1)^{4/l}}\right)^{(p-1)/4}, & \text{if } l \text{ is even,} \\ 2p\left(\dfrac{p-1}{8(2^{l}-1)^{2/l}}\right)^{(p-1)/4}, & \text{if } l \text{ is odd.} \end{cases}$$

From that we obtain $h_1(p) \leqslant 2p((p-1)/31.997158\ldots)^{(p-1)/4}$ which is better than Carlitz's and Metsänkyla's results. For the fields $Q(\zeta_{2^n})$ and $Q(\zeta_{p^n})$ $(n \geqslant 2)$, we get the similar results.

Let p be an odd prime, $\zeta_p = e^{(2\pi i)/p}$, $h(p)$ and $h_2(p)$ be the class number of the cyclotomic field $Q(\zeta_p)$ and its maximal real subfield $Q(\zeta_p + \zeta_p^{-1})$ respectively. In 1850 Kummer [2] proved that $h_1(p) = h(p)/h_2(p)$ is a rational integer which is called the first factor of the class number of $Q(\zeta_p)$. Kummer conjectured that

$$h_1(p) \sim 2p\left(\frac{p}{4\pi^2}\right)^{(p-1)/4} = 2p\left(\frac{p}{39.4784\ldots}\right)^{(p-1)/4} \quad (\text{when } p \to \infty).$$

In 1961 L. Carlitz [1] proved that $h_1(p) < (p-1)((p-1)/2)^{(p-1)/4}$. In 1974 T. Metsänkyla proved that $h_1(p) < 2p((p-1)(p-2)/24p)^{(p-1)/4}$. In this short paper I will show that

$$h_1(p) < 2p\left(\frac{p-1}{31.997158\ldots}\right)^{(p-1)/4}. \tag{1}$$

Exactly speaking, I will show the following theorem.

THEOREM. *Let l be the order of* 2 (mod p). *Then*

$$h_1(p) \leqslant \begin{cases} 2p\left(\dfrac{p-1}{8(2^{l/2}+1)^{4/l}}\right)^{(p-1)/4}, & \textit{if } l \textit{ is even,} \\ 2p\left(\dfrac{p-1}{8(2^{l}-1)^{2/l}}\right)^{(p-1)/4}, & \textit{if } l \textit{ is odd.} \end{cases}$$

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FROM THE THEOREM TO (1). For even l, from the theorem we know that

$$h_1(p) \leqslant 2p\left(\frac{p-1}{8(2^{l/2}+1)^{4/l}}\right)^{(p-1)/4} \leqslant 2p\left(\frac{p-1}{32}\right)^{(p-1)/4}.$$

For odd l, it is easy to see that if $p > 127$ then $l \geqslant 11$ and

$$(2^l-1)^{2/l} \geqslant (2^{11}-1)^{2/11} = 3.9996448\ldots.$$

Thus

$$h_1(p) \leqslant 2p\left(\frac{p-1}{8\times 3.9996448\ldots}\right)^{(p-1)/4} = 2p\left(\frac{p-1}{31.997158}\right)^{(p-1)/4}.$$

For $p \leqslant 127$ we can verify (1) directly.

To prove the theorem we need

LEMMA. *Let χ be an odd Dirichlet character with the conductor f. If $f \not\equiv 2 \pmod 4$, then*

$$\sum_{1\leqslant a<f} \chi(a)a = \frac{f}{\bar{\chi}(2)-2} \sum_{1\leqslant a<f/2} \chi(a).$$

PROOF. Long [3] proved this lemma for quadratic character χ. His method is available for any character satisfying the conditions of the lemma. We restate it for sake of completeness.

When $2 \nmid f$ then, on one hand,

$$\begin{aligned} -\sum_{0<a<f} \chi(a)a &= -\sum_{\substack{0<a<f\\ 2|a}} (\chi(a)a + \chi(f-a)(f-a)) \\ &= -\sum_{0<a<f/2} \chi(2a)2a + \sum_{0<a<f/2} \chi(2a)(f-2a) \\ &= \chi(2) \sum_{0<a<f/2} (f-4a)\chi(a). \end{aligned}$$

Thus

$$-\bar{\chi}(2) \sum_{0<a<f} \chi(a)a = \sum_{0<a<f/2} (f-4a)\chi(a). \tag{2}$$

On the other hand,

$$\begin{aligned} -\sum_{0<a<f} \chi(a)a &= -\sum_{0<a<f/2} \chi(a)a - \sum_{0<a<f/2} \chi(f-a)(f-a) \\ &= -\sum_{0<a<f/2} \chi(a)a + \sum_{0<a<f/2} \chi(a)(f-a) \qquad (3) \\ &= \sum_{0<a<f/2} (f-2a)\chi(a). \end{aligned}$$

From (2) and (3) we obtain

$$\sum_{0<a<f} \chi(a)a = \frac{f}{\bar{\chi}(2)-2} \sum_{0<a<f/2} \chi(a).$$

When $4 \mid f$ then $\chi(2) = 0$ and $\chi(x + f/2) = -\chi(x)$. Thus

$$\begin{aligned}\sum_{0<a<f} \chi(a)a &= \sum_{0<a<f/2} \chi(a)a + \sum_{0<a<f/2} \chi\left(a + \frac{f}{2}\right)\left(a + \frac{f}{2}\right) \\ &= \sum_{0<a<f/2} \chi(a)a - \sum_{0<a<f/2} \chi(a)\left(a + \frac{f}{2}\right) \\ &= -\frac{f}{2} \sum_{0<a<f/2} \chi(a).\end{aligned}$$

PROOF OF THE THEOREM. Starting from the analytic formula of $h_1(p)$,

$$\begin{aligned}h_1(p) &= 2p \prod_{\chi(-1)=-1} \sum_{a=1}^{p-1} \left(-\frac{1}{2p}\right)\chi(a)a \\ &= (2p)^{-(p-1)/2+1} \prod_{\chi(-1)=1} \left|\sum_{a=1}^{p-1} \chi(a)a\right|,\end{aligned}$$

where χ is taken to be the $(p-1)/2$ odd characters of mod p. Let g be a primitive root of mod p and $\chi_0(a) = \exp((2\pi i)/(p-1) \cdot \operatorname{ind}_g a)$ $(1 \leqslant a \leqslant p-1)$. Then

$$h_1(p) = (2p)^{-(p-1)/2+1} \prod_{\lambda=1}^{(p-1)/2} |\chi_0^{2\lambda-1}(a)a|.$$

From the above lemma we also have

$$(4)\quad h_1(p) = (2p) \cdot 2^{-(p-1)/2}\left(\prod_{\lambda=1}^{(p-1)/2} |2 - \chi_0^{2\lambda-1}(2)|^{-1}\right) \prod_{\lambda=1}^{(p-1)/2} \left|\sum_{a=1}^{(p-1)/2} \chi_0^{2\lambda-1}(a)\right|.$$

Using the orthogonal relations of odd characters of mod p,

$$\begin{aligned}\sum_{\lambda=1}^{(p-1)/2} \chi_0^{2\lambda-1}(a)\bar{\chi}_0^{2\lambda-1}(b) &= \sum_{\lambda=1}^{(p-1)/2} \exp\left(\frac{2\pi i \cdot \operatorname{ind}_g\left(\frac{a}{b}\right)}{p-1} \cdot (2\lambda - 1)\right) \\ &= \begin{cases} (p-1)/2, & \text{if } a \equiv b \pmod{p}, \\ -(p-1)/2, & \text{if } a \equiv -b \pmod{p}, \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

For $1 \leqslant a, b \leqslant p-1$, we have

$$\begin{aligned}\prod_{\lambda=1}^{(p-1)/2} \left|\sum_{a=1}^{(p-1)/2} \chi_0^{2\lambda-1}(a)\right|^{4/(p-1)} &= \prod_{\lambda=1}^{(p-1)/2} \left(\sum_{a,b=1}^{(p-1)/2} \chi_0^{2\lambda-1}(a)\bar{\chi}_0^{2\lambda-1}(b)\right)^{2/(p-1)} \\ &\leqslant \frac{2}{p-1} \sum_{\lambda=1}^{(p-1)/2} \sum_{a,b=1}^{(p-1)/2} \chi_0^{2\lambda-1}(a)\bar{\chi}_0^{2\lambda-1}(b) \\ &= \frac{2}{p-1} \sum_{a,b=1}^{(p-1)/2} \sum_{\lambda=1}^{(p-1)/2} \chi_0^{2\lambda-1}(a)\bar{\chi}_0^{2\lambda-1}(b) \\ &= \frac{2}{p-1} \cdot \frac{p-1}{2} \cdot \frac{p-1}{2} = \frac{p-1}{2}.\end{aligned}$$

Thus

$$\prod_{\lambda=1}^{(p-1)/2}\left|\sum_{a=1}^{(p-1)/2}\chi_0^{2\lambda-1}(a)\right| \leqslant \left(\frac{p-1}{2}\right)^{(p-1)/4}. \tag{5}$$

On the other hand, we can assume $\operatorname{ind}_g 2 = (p-1)/l$ without loss of generality. So

$$\begin{aligned}\prod_{\lambda=1}^{(p-1)/2}\left(2-\chi_0^{2\lambda-1}(2)\right) &= \prod_{\lambda=1}^{(p-1)/2}\left(2-\exp\left(\frac{2\pi i\cdot \operatorname{ind}_g 2}{p-1}(2\lambda-1)\right)\right)\\ &= \prod_{\lambda=1}^{(p-1)/2}\left(2-\exp\left(\frac{2\pi i(2\lambda-1)}{l}\right)\right)\end{aligned}$$

(6)

$$= \begin{cases} \left(\prod_{\mu=0}^{l-1}(2-e^{2\pi i\mu/l})\right)^{(p-1)/2l} = (2^l-1)^{(p-1)/2l}, & \text{if } l \text{ is odd,}\\ \left(\dfrac{\prod_{\mu=0}^{l-1}(2-e^{2\pi i\mu/l})}{\prod_{\mu=0}^{l/2-1}(2-e^{2\pi i\mu/(l/2)})}\right)^{(p-1)/l} = \left(\dfrac{2^l-1}{2^{l/2}-1}\right)^{(p-1)/l} & \\ \qquad = (2^{l/2}+1)^{(p-1)/l}, & \text{if } l \text{ is even.}\end{cases}$$

Combining (4), (5), and (6) we obtain the theorem.

REMARK. Let $h_1(p^n)$ be the first factor of class number of field $Q(\zeta_{p^n})$. Using the similar argument and the same lemma we can get that

(i) $h_1(2^n) \leqslant 2^{n+(n-6)2^{n-3}}$ $(n \geqslant 2)$,

(ii) For odd prime p

$$h_1(p^n) \leqslant \begin{cases} 2p^n\left(\dfrac{\varphi(p^n)}{8(2^{l/2}+1)^{4/l}}\right)^{\varphi(p^n)/4}, & \text{if } l \text{ is even,}\\ 2p^n\left(\dfrac{\varphi(p^n)}{8(2^l-1)^{2/l}}\right)^{\varphi(p^n)/4}, & \text{if } l \text{ is odd,}\end{cases}$$

where l is the order of 2 (mod p^n).

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