SPHERICAL HARMONICS GENERATING BOUNDED BIHARMONICS

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Abstract. Let $H^2B(R)$ be the family of bounded nonharmonic biharmonic functions on a Riemannian manifold $R$. On the punctured Euclidean $N$-space $E^N_0 = \{x = (x^1, \ldots, x^N) | 0 < |x| < \infty\}$, $H^2B$ is void for $N > 3$, whereas for $N = 2, 3$, it is generated by certain fundamental spherical harmonics. It is also known that $H^2B$ remains void on the Riemannian manifold $E^N_0$, $N > 3$, obtained by endowing $E^N_0$ with the non-Euclidean metric $ds_a = r^a |dx|$, $a \in \mathbb{R}$.

The purpose of the present paper is to show that the fundamental spherical harmonics continue generating $H^2B(E^N_a)$, despite the distorting metric $ds_a$. An analogous result holds for $E^2_a$.

Let $H^2B(R)$ be the family of bounded nonharmonic biharmonic functions on a Riemannian manifold $R$. It was shown in Sario and Wang [2] that on the punctured Euclidean $N$-space $E^N_0 = \{x = (x^1, \ldots, x^N) | 0 < |x| < \infty\}$, $H^2B$ is void for $N > 3$, whereas for $N = 2, 3$, it is generated by certain fundamental spherical harmonics.

In Sario and Wang [3], it was proved that $H^2B$ remains void on the Riemannian manifold $E^N_a$, $N > 3$, obtained by endowing $E^N_0$ with the non-Euclidean metric $ds_a = r^a |dx|$, $a \in \mathbb{R}$. The cases $N = 2, 3$ were left open.

The purpose of the present paper is to show that the fundamental spherical harmonics continue generating $H^2B(E^N_a)$, despite the distorting metric $ds_a$. An analogous result holds for $E^2_a$.

1.1. Consider the space

$$E^N_a = \{x = (r, \theta^1, \ldots, \theta^{N-1}) | 0 < r < \infty, ds_a = r^a |dx|, a \in \mathbb{R}\}$$

with polar coordinates $(r, \theta) = (r, \theta^1, \ldots, \theta^{N-1})$. We first state some facts from [3].

**Lemma 1.** There exist no $H^2B$ functions on $E^N_1$ for $N > 2$.

Thus when we are searching for generators of $H^2B$ functions, we may ignore the case of $\alpha = -1$.

**Lemma 2.** For any $\alpha \neq -1$, and $N \geq 2$, every $h \in H(E^N_a)$ has an expansion

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm}r^p + b_{nm}r^{q})S_{nm} + a\sigma(r) + b,$$

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where

\[ \sigma(r) = \begin{cases} \log r, & \text{if } N = 2, \\ r^{-(N-2)(\alpha+1)}, & \text{if } N > 2. \end{cases} \]

**Lemma 3.** Let \( u \) be a biharmonic function on \( E_\alpha^N \), \( N \geq 2, \alpha \neq -1 \), with

\[ \Delta u = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm}r^{p_n} + b_{nm}r^{q_n})s_{nm} + a\sigma(r) + b. \]

Then

\[
\begin{align*}
-u &= \frac{-1}{4(\alpha + 1)} \left[ \sum_{n \neq \mu} \sum_{m=1}^{m_n} \frac{1}{P_n} a_{nm}r^{p_n+2\alpha+2}s_{nm} + \sum_{n \neq \nu} \sum_{m=1}^{m_n} \frac{1}{Q_n} b_{nm}r^{q_n+2\alpha+2}s_{nm} \right] \\
&\quad - \frac{1}{2(\alpha + 1)} r^{2(N+2)(\alpha+1)} \log r \left( \sum_{m=1}^{m_n} a_{nm}s_{nm} + \sum_{m=1}^{m_n} b_{nm}s_{nm} \right) \\
&\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm}r^{p_n} + d_{nm}r^{q_n})s_{nm} + a\tau(r) + b\sigma(r) + c\sigma(r) + d,
\end{align*}
\]

where

\[ p_n = \frac{1}{2} \left\{ -(N-2)(\alpha+1) + \left[ (N-2)(\alpha+1)^2 + 4n(n+N-2) \right]^{1/2} \right\}, \]

\[ q_n = \frac{1}{2} \left\{ -(N-2)(\alpha+1) - \left[ (N-2)(\alpha+1)^2 + 4n(n+N-2) \right]^{1/2} \right\}, \]

\[ P_n = \frac{1}{2} N(\alpha+1) + p_n, \quad Q_n = \frac{1}{2} N(\alpha+1) + q_n, \]

\( \mu \) and \( \nu \) are defined by \( P_\mu = 0 \) and \( Q_\nu = 0 \) respectively, \( \sigma(r) \) is as above,

\[ s(r) = \frac{-1}{2N(\alpha+1)^2} r^{2\alpha+2}, \]

and

\[
\tau(r) = \begin{cases} 
\frac{s(r)\left[ \log r - \frac{1}{\alpha+1} \right]}{2\alpha+2}, & \text{if } N = 2, \\
\frac{1}{2(N-4)(\alpha+1)^2} r^{-(N-4)(\alpha+1)}, & \text{if } N \neq 2, 4.
\end{cases}
\]

1.2. **Generators of \( H^2B \)-functions on \( E_\alpha^2 \).** We claim

**Theorem 1.** \( H^2B(E_\alpha^2) \), where \( \alpha = -1 \pm k/2, k \in \mathbb{Z}^+ \), is generated by 1, \( \sin k\theta \), and \( \cos k\theta \).
PROOF. We shall establish the theorem for the case $\alpha = -1 - k/2$, the other case being entirely similar. Let $S_{n1} = \cos n\theta$ and $S_{n2} = \sin n\theta$. From Lemma 3 we calculate

\[ p_n = \frac{1}{2} \left\{ -(N-2)(\alpha + 1) + \left[ (N-2)^2(\alpha + 1)^2 + 4n(n+N-2) \right]^{1/2} \right\} = \frac{1}{2} \left[ (4n^2) \right]^{1/2} = n, \]

\[ q_n = \frac{1}{2} \left\{ -(N-2)(\alpha + 1) - \left[ (N-2)^2(\alpha + 1)^2 + 4n(n+N-2) \right]^{1/2} \right\} = -\frac{1}{2} \left[ (4n^2) \right]^{1/2} = -n. \]

$P_n = 0$ implies $\frac{1}{2}N(\alpha + 1) + p_n = 0$, which yields $n = \frac{k}{2}$. Thus $\mu = \frac{k}{2}$ if $k$ is even. Since $\mu$ must be an integer, there is no $\mu$ term if $k$ is odd.

$Q_n = 0$ gives $\frac{1}{2}N(\alpha + 1) + q_n = 0$, whence $-n = \frac{k}{2}$, which is impossible, since both $n$ and $k$ are positive. Thus there is no $\nu$ term.

We may represent $u$ by

\[ u = \sum_{n \neq k/2} \sum_{m=1}^{2} A_{nm} r^{-\nu-k} S_{nm} + \sum_{n=1}^{\infty} \sum_{m=1}^{2} B_{nm} r^{-\nu-k} S_{nm} + r^{-k/2} \log r \cdot \sum_{m=1}^{2} a_{(k/2)m} S_{(k/2)m} + \sum_{n=1}^{\infty} \sum_{m=1}^{2} (c_{nm} r^n + d_{nm} r^{-n}) S_{nm} + a\tau(r) + b\sigma(r) + c\phi(r) + d. \]

We wish to show that all coefficients are zero except possibly $A_{k1}, A_{k2}$ and $d$.

Suppose $c_{nm} \neq 0$ for some $(n, m)$. Choose $\rho(r) \in C^\infty_0$, $\rho \geq 0$, supp $\rho \subset (0, 1)$. Define $\varphi_t(r) = \rho(r - t)$ for $0 < t < \infty$, and $\varphi_t(r, \theta) = \rho, S_{nm}$. Then supp $\varphi_t \subset (t, t + 1)$, and by the orthogonality of the $(S_{nm})$,

\[ (u, \varphi_t) = \int_{E_n^2} u \rho, S_{nm} r^{-\nu-k} - k \ d\theta \ dr \]

\[ = c \int_t^{t+1} \left[ A_{nm} r^{-\nu-k} + B_{nm} r^{-\nu-k} + a_{(k/2)m} r^{-k/2} \log r + c_{nm} r^n + d_{nm} r^{-n} \right] \rho_t r^{-k-1} \ dr, \]

where the third term in the integrand appears only if $n = k/2$; also,

\[ (1, |\varphi_t|) = c \int_t^{t+1} \rho_t r^{-k-1} \ dr, \]

the $c$'s denoting constants, not necessarily the same.

Certainly $\varphi_t \in L^1(E_n^2)$, and thus we have the inequality

\[ |(u, \varphi_t)| \leq (\sup |u|)(1, |\varphi_t|). \]

But as $t \to \infty$, $(u, \varphi_t) \sim ct^{\nu-k-1}$ and $(1, |\varphi_t|) \sim ct^{-k-1}$, thus forcing $c_{nm} = 0$ for all $n$ and $m$. 

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If $A_{nm} \neq 0$ for $n > k$, the same estimate with the same test function as in the first step gives, as $t \to \infty$,

$$(u, \varphi_r) \sim ct^{n-2k-1}, \quad (1, |\varphi_r|) = O(t^{-k-1}).$$

Therefore $A_{nm} = 0$ for $n > k$.

Now suppose $B_{nm} \neq 0$ for some $(n, m)$. Choose $\rho(r) \in C_0^\infty$, $\rho \geq 0$, $\text{supp} \rho \subset (\beta, \gamma)$, $0 < \beta < \gamma < 1$. Define $\rho_t(r) = \rho(r/t)$ for $0 < t < 1$, and $\varphi_t(r, \theta) = \varphi_1 S_{nm}$.

Then

$$(u, \varphi_r) = c \int_{\beta t}^{\gamma t} \left[ A_{nm} r^{-n-k} + B_{nm} r^{-n-k} + a_{(k/2)m} r^{-k/2} \log r + d_{nm} r^{-n} \right] \rho_t r^{-k-1} dr,$$

where the first term in the integrand appears only if $n < k$, and the third only if $n = k/2$. On the other hand,

$$(1, |\varphi_r|) = c \int_{\beta t}^{\gamma t} \rho_t r^{-k-1} dr.$$ \hspace{1cm} (As $t \to 0$,)

$$(u, \varphi_r) \sim ct^{-n-2k}, \quad (1, |\varphi_r|) = O(t^{-k}),$$

and again we have a contradiction to $\varphi_r \in L^1(E^2_\beta)$. Therefore, $B_{nm} = 0$ for all $m$ and $n$.

Suppose $d_{nm} \neq 0$ for some $n \geq k$. Using the same test function as above we obtain

$$(u, \varphi_r) = c \int_{\beta t}^{\gamma t} d_{nm} r^{-n-k} \rho_t r^{-k-1} dr \sim ct^{-n-k}, \hspace{1cm} \text{as } t \to 0,$$

$$(1, |\varphi_r|) = c \int_{\beta t}^{\gamma t} \rho_t r^{-k-1} dr = O(t^{-k}), \hspace{1cm} \text{as } t \to 0.$$ \hspace{1cm} (A fortiori, $d_{nm} = 0$ for $n \geq k$.)

We have reduced $u$ to

$$u = A_{k1} S_{k1} + A_{k2} S_{k2} + d + \sum_{n=1}^{k-1} \sum_{m=1}^{2} A_{nm} r^{-n-k} S_{nm}$$

$$+ r^{-k/2} \log r \sum_{m=1}^{2} a_{(k/2)m} S_{(k/2)m} + \sum_{n=1}^{k-1} \sum_{m=1}^{2} d_{nm} r^{-n} S_{nm}$$

$$+ a \tau(r) + b s(r) + c \sigma(r).$$

Since $u$ is bounded, we have, on letting $r \to \infty$, $A_{nm} = d_{nm} = 0$ for $n = 1, \ldots, k - 1$, and $a_{(k/2)1} = a_{(k/2)2} = a = b = c = 0$. We infer that

$$u = A_{k1} S_{k1} + A_{k2} S_{k2} + d = A_{k1} \cos k \theta + A_{k2} \sin k \theta + d.$$ \hspace{1cm} (The proof of Theorem 1 is complete.)

In the case $k = 2$, we have $\alpha = 0$, and therefore

**Corollary [2].** $H^2 B(E^2_0)$ is generated by $1$, $\sin 2 \theta$, and $\cos 2 \theta$.\hspace{1cm}
1.3. Generators of $H^2B$ functions on $E^3_\alpha$. We proceed to show

**Theorem 2.** If $\alpha = -1 \pm \sqrt{\frac{1}{2}k(k + 1)}$, $k \in \mathbb{Z}^+$, then $H^2B(E^3_\alpha)$ is generated by 1 and $(S_{km})_{m=1}^{\infty}$, the set of fundamental spherical harmonics of order $k$.

**Proof.** Again we will give the proof only for $\alpha = -1 - \sqrt{\frac{1}{2}k(k + 1)}$, the other case being similar. We start with

$$p_n = \frac{1}{2} \left\{ -(3 - 2)(\alpha + 1) + \left[ (3 - 2)(\alpha + 1)^2 + 4n(n + 3 - 2) \right]^{1/2} \right\},$$

$$q_n = \frac{1}{2} \left\{ \sqrt{2k(k + 1)} + \sqrt{2k(k + 1) + 16n(n + 1)} \right\}.$$  

If $P_n = 0$, then $\frac{1}{2}N(\alpha + 1) + p_n = 0$ or $n(n + 1) = \frac{3}{8}k(k + 1)$. For fixed $k$ this is a quadratic in $n$ which has at most one positive integral solution. If such a solution exists, we denote it by $n = \mu$. Observe that $\mu < k$.

If $Q_n = 0$, then

$$-\frac{3}{4}\sqrt{2k(k + 1)} = -\frac{3}{4}\sqrt{2k(k + 1) + \frac{1}{2}} \sqrt{2k(k + 1) + 16n(n + 1)},$$

which has no solution, so that there is no $\nu$ term in $u \in H^2B(E^3_\alpha)$.

Thus $u$ has the representation

$$u = \sum_{n \neq \mu} \sum_{m=1}^{m_n} A_{nm} r^{p_n + 2a + 2} S_{nm} + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} B_{nm} r^{q_n + 2a + 2} S_{nm}$$

$$+ r^{(2-N/2)(\alpha+1)} \log r \cdot \sum_{m=1}^{m_n} a_{nm} S_{nm} + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm} r^{p_n} + d_{nm} r^{q_n}) S_{nm}$$

$$+ a\tau(r) + b\sigma(r) + c.$$

The remainder of the proof will be divided into Steps I–V.

I. For any given $n$, the highest order term is obviously $c_{nm} r^{p_n}$; we eliminate this first.

Choose $\rho \in C_0^\infty$, $\rho \geq 0$, supp $\rho \subset (0, 1)$. Define $\rho_t(r) = \rho(r - t)$, and $\varphi_t = \rho_t S_{nm}$.

Then

$$(u, \varphi_t) = c \int_{t}^{t+1} \left[ A_{nm} r^{p_n + 2a + 2} + B_{nm} r^{q_n + 2a + 2} + a_{nm} r^{-1/4(2k(k + 1))^{1/2}} \log r + c_{nm} r^{p_n} + d_{nm} r^{q_n} \right]$$

$$\cdot \rho_t r^{-1-3/2(2k(k + 1))^{1/2}} dr,$$

$$\left(1, \varphi_t \right) = c \int_{t}^{t+1} \rho_t r^{-1-3/2(2k(k + 1))^{1/2}} dr = O(t^{-1-3/2(2k(k + 1))^{1/2}}),$$

whereas

$$(u, \varphi_t) \sim c t^{p_n-1-3/2(2k(k + 1))^{1/2}}, \quad \text{as } t \to \infty.$$  

Since $p_n > 0$ for all $n$, it follows that $c_{nm} = 0$ for all $n$ and $m$. 

II. For $n > k$, $p_n + 2\alpha + 2 > 0$, $p_n + 2\alpha + 2 > q_n$, and $p_n + 2\alpha + 2 > \mu$. After Step I above, the latter statements assure us that, for a fixed $n > k$, $A_{nm} r^{p_n+2\alpha+2}$ is the dominant term as $r \to \infty$. If $A_{nm} \neq 0$ for $n > k$, the test function from Step I gives
\[
(u, \varphi_t) = \int_1^{t+1} |A_{nm} r^{p_n+2\alpha+2} + \ldots| \rho_t r^{-1-3/2(2(k(k+1))^1/2} dr
\]
\[
\sim c t^{p_n+2\alpha+2-1-3/2(2(k(k+1))^1/2}, \quad \text{as } t \to \infty.
\]
Thus $A_{nm} = 0$ for $n > k$.

III. Suppose $B_{nm} \neq 0$ for some $n$. Choose $\rho \in C_0$, $\rho \geq 0$, supp $\rho \subset (\beta, \gamma)$, $0 < \beta < \gamma < 1$. Define $\rho_t(r) = \rho(r/t)$, $\varphi_t = \rho_t S_{nm}$. Then
\[
(u, \varphi_t) = c \int_{\beta t}^{t+1} |B_{nm} r^{q_n+2\alpha+2} + \ldots| \rho_t r^{-1-3/2(2(k(k+1))^1/2} dr
\]
\[
\sim c t^{q_n+2\alpha+2-3/2(2(k(k+1))^1/2}, \quad \text{as } t \to 0,
\]
\[
(1, \varphi_t) = c \int_{\beta t}^{t+1} \rho_t r^{-1-3/2(2(k(k+1))^1/2} dr = O(t^{-3/2(2(k(k+1))^1/2}), \quad \text{as } t \to 0.
\]
The relation $q_n + 2\alpha + 2 < 0$ for all $n$ implies $B_{nm} = 0$ for all $n$ and $m$.

IV. As in Step II, assume $n > k$. In view of the above, the test function from Step III gives
\[
(u, \varphi_t) = c \int_{\beta t}^{t+1} d_{nm} r^{q_n} \rho_t r^{-1-3/2(2(k(k+1))^1/2} dr \sim c t^{q_n-3/2(2(k(k+1))^1/2}
\]
and
\[
(1, \varphi_t) = c \int_{\beta t}^{t+1} \rho_t r^{-1-3/2(2(k(k+1))^1/2} dr = O(t^{-3/2(2(k(k+1))^1/2}), \quad \text{as } t \to 0.
\]
Inasmuch as $q_n < 0$ for all $n$, $d_n = 0$ for all $n > k$.

V. We have now reduced $u$ to
\[
u = \sum_{n=1}^{k} \sum_{n \neq m} A_{nm} r^{p_n+2\alpha+2} S_{nm} + r^{-1/4(2(k(k+1))^1/2} \log r \cdot \sum_{m=1}^{m} a_{nm} S_{nm}
\]
\[
+ \sum_{n=1}^{k} \sum_{m=1}^{m} d_{nm} r^{q_n} S_{nm} + a \tau(r) + bs(r) + ca(r) + d.
\]
As in the proof of Theorem 1, we may eliminate a finite number of linearly independent unbounded terms and are left with
\[
u = \sum_{m=1}^{m} A_{km} S_{km} + d.
\]
The proof of Theorem 2 is complete.

In particular, when $k = 1$, we have $\alpha = 0$, and the $S_{1m}$ are cos $\theta$ cos $\psi$, sin $\theta$ sin $\psi$ and cos $\theta$, where $\theta$ is the angle between the vector and the $z$-axis.

**Corollary [2].** $HB^2(E_0^2)$ is generated by $1$, cos $\theta$ cos $\psi$, sin $\theta$ sin $\psi$, and cos $\theta$. 
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