ON A CHARACTERIZATION OF INVARIANT SUBSPACE LATTICES OF WEIGHTED SHIFTS

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Abstract. The paper concerns itself with the characterization of invariant subspace lattices of weighted shift operators on the Hilbert space $l^2$ with suitable conditions on their weights. This characterization is also extended to the case of Banach spaces $l^p$, $1 < p < \infty$.

1. Introduction. Let $l^2$ be the Hilbert space of all square-summable complex sequences $x = \{x_0, x_1, x_2, \ldots\}$ with the norm

$$\|x\| = \left( \sum_{m=0}^{\infty} |x_m|^2 \right)^{1/2}.$$

If $\{w_m\}_{m=0}^{\infty}$ is a bounded sequence of nonzero complex numbers, then the operator $T$, defined by

$$T \{x_0, x_1, x_2, \ldots\} = \{0, w_0 x_0, w_1 x_1, w_2 x_2, \ldots\},$$

is called a unilateral (forward) weighted shift on $l^2$ with the weight sequence $\{w_m\}_{m=0}^{\infty}$. We may, and shall, assume without any loss of generality that the weights $w_m$ are positive real numbers [2, Problem 2]. By an invariant subspace $M$ of $T$ we shall mean a closed linear manifold of $l^2$ such that $TM \subseteq M$. We shall denote by $\text{Lat} \, T$ the lattice of invariant subspaces of $T$. Various authors have characterized $\text{Lat} \, T$ under suitable conditions on the weight sequence $\{w_m\}_{m=0}^{\infty}$. The following characterization is due to Nikolskii [3]:

If the weight sequence $\{w_m\}_{m=0}^{\infty}$ is monotonically decreasing to zero and belongs to $l^2$, then

$$\text{Lat} \, T = \{ \{0\}, M_1, M_2, \ldots, M_k, \ldots, l^2 \},$$

where

$$M_k = \{ x \in l^2 : x_m = 0, m < k \}.$$

The particular case of this result in which $w_m = 2^{-m}$ was originally obtained by Donoghue [1]; see also [5, p. 66]. The central theme of this paper is to exhibit that Nikolskii's theorem holds under more general conditions on the weight sequence...
$\{w_m\}_{m=0}^{\infty}$. We shall say that the sequence $\{w_m\}_{m=0}^{\infty}$ is of bounded variation if

$$
\sum_{m=0}^{\infty} |w_m - w_{m+1}| < \infty.
$$

It is easy to see that if $\{w_m\}_{m=0}^{\infty}$ is monotonically decreasing, then it is of bounded variation, but the converse is not true.

2. We now prove

**Theorem 1.** If the weight sequence $\{w_m\}_{m=0}^{\infty}$ is of bounded variation and

$$
\delta = \sup_{m \geq 2, n} \sum_{k=0}^{\infty} \left( \frac{w_{k+m} \cdots w_{k+n}}{w_m w_{m+1} \cdots w_n} \right)^2 < \infty,
$$

then $\text{Lat } T$ is given by (1).

**Proof.** Let $\{e_m\}_{m=0}^{\infty}$ be the standard orthonormal basis of $l^2$ and let $M$ be an invariant subspace of $T$. Firstly, we proceed to show that if a vector $x = \{x_m\}_{m=0}^{\infty}$, with $x_0 \neq 0$, is in $M$, then $M = l^2$. As

$$
T^n x = \left\{ 0, 0, \ldots, 0, x_0 w_0 w_1 \cdots w_{n-1}, x_1 w_1 w_2 \cdots w_n, \ldots \right\},
$$

it follows that

$$
\left\| \frac{T^n x}{x_0 w_0 w_1 \cdots w_{n-1}} - e_n \right\|^2 = \sum_{m=0}^{\infty} \left( \frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_n} \right)^2 \left| \frac{x_{m+1}}{x_0} \right|^2
$$

$$
= \frac{w_n^2}{w_0^2 |x_0|^2} \sum_{m=0}^{\infty} \left( \frac{w_{m+1} \cdots w_{m+n}}{w_1 \cdots w_n} \right)^2 |x_{m+1}|^2
$$

$$
\leq \frac{w_n^2 \|x\|^2}{w_0^2 |x_0|^2} \sum_{m=0}^{\infty} \left( \frac{w_{m+1} \cdots w_{m+n}}{w_1 \cdots w_n} \right)^2
$$

$$
= \frac{w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} \left( \frac{w_{m+2} \cdots w_{m+n}}{w_2 \cdots w_n} \right)^2 w_{m+1}^2
$$

$$
= \frac{w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( \frac{w_{k+2} \cdots w_{k+n}}{w_2 \cdots w_n} \right)^2 \left( w_{m+1}^2 - w_{m+2}^2 \right)
$$

(by Abel's transformation [7])

$$
\lesssim \frac{\delta w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} \left( w_{m+1}^2 - w_{m+2}^2 \right) \quad \text{(by (2))}
$$

$$
\lesssim \frac{\delta w_n^2 \|x\|^2}{w_0^2 w_1^2 |x_0|^2} \sum_{m=0}^{\infty} \left| w_{m+1} - w_{m+2} \right| \left( w_{m+1} + w_{m+2} \right)
$$

$$
\lesssim 2 \delta w_n^2 \|x\|^2 \sum_{m=0}^{\infty} \left| w_{m+1} - w_{m+2} \right| \leq C w_n^2,
$$

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where $\mu = \sup_m w_m$ and $C$ is a constant. Since \( \{e_n\}_{n=0}^{\infty} \) is an orthonormal basis in $l^2$ and $\sum_{n=0}^{\infty} w_n^2 < \infty$, it follows by the Paley-Wiener theorem [6, p. 208] that the system

$$\begin{bmatrix}
T^n x \\
x_0 w_0 w_1 \cdots w_{n-1}
\end{bmatrix}
$$

is a Riesz basis in $l^2$, whence we conclude that $M = l^2$. Again, if $x_0 = 0$ and $k$ is the least natural number such that $x_k \neq 0$, then we can similarly show that

$$\bigvee_{n=0}^{\infty} \{ T^n x \} = M_k.$$

Thus we have shown that every cyclic subspace of $T$ is an $M_k$. Now our theorem follows by observing that the span of any number of $M_k$'s is again an $M_k$.

Our next theorem, which we state without proof, shows that the condition of bounded variation can be dispensed with, provided that condition (2) is replaced by a more stringent condition; even so, our theorem generalizes Nikolskii’s result.

**Theorem 2.** If the weight sequence $\{ w_m \}_{m=0}^{\infty}$ satisfies the condition

$$\delta = \sup_{m \geq 1, n} \sum_{k=0}^{\infty} \left( \frac{w_{k+m} \cdots w_{k+n}}{w_{m} w_{m+1} \cdots w_{n}} \right)^2 < \infty,$$

then Lat $T$ is given by (1).

Now we extend Theorem 2 for the $l^p$ spaces, $1 < p < \infty$, which, in turn, generalizes Nikolskii’s main result [3, Theorem 2]. We shall denote by $q$ the Hölder conjugate of $p$, i.e., the number determined by $1/p + 1/q = 1$.

**Theorem 3.** Let $T$ be a unilateral (forward) weighted shift on $l^p$ with weight sequence $\{ w_m \}_{m=0}^{\infty}$ and let

$$\delta = \sup_{m \geq 1, n} \sum_{k=0}^{\infty} \left( \frac{w_{k+m} \cdots w_{k+n}}{w_{m} w_{m+1} \cdots w_{n}} \right)^q < \infty.$$

Then Lat $T$ is given by

$$\text{Lat} T = \{ \{0\}, M_1, M_2, \ldots, M_k, \ldots, l^p \},$$

where

$$M_k = \{ x \in l^p : x_m = 0, m < k \}.$$

We shall only sketch the proof of this theorem. Let $M$ be any subspace of $l^p$ invariant under $T$. If a vector $x = \{ x_m \}_{m=0}^{\infty}$, $x_0 \neq 0$, is in $M$, we intend to show that $M = l^p$. Let $y = \{ y_m \}_{m=0}^{\infty}$ be any element in $l^q$ such that $y(T^n x) = 0$, $n = 0, 1, 2, \ldots$. It will suffice to show that $y = 0$. We have, for $x_0 = 1$,

$$y_n = \frac{-1}{w_0 w_1 \cdots w_{n-1}} \sum_{m=0}^{\infty} w_{m+1} w_{m+2} \cdots w_{m+n} x_{m+1} y_{m+n+1},$$
and hence, using Hölder’s inequality, Abel’s transformation and condition (4) in succession, we obtain

\[
\left| y_n \right| \leq \frac{w_n}{w_0} \left( \sum_{m=0}^{\infty} \left| x_{m+1} \right|^p \right)^{1/p} \left( \sum_{m=0}^{\infty} \left( \frac{w_{m+1} \cdots w_{m+n}}{w_1 w_2 \cdots w_n} \right) q \left| y_{m+n+1} \right|^q \right)^{1/q}
\]

This inequality is the main step in the proof of Nikolskii’s theorem [3]. By following exactly the line of his argument, we can show that \( y = 0 \), and we are done.

If \( x = \{x_m\}_{m=0}^{\infty} \) is a vector in \( M \) with \( x_0 = 0 \), let \( k \) be the least positive integer such that \( x_k \neq 0 \). By repeating the argument used above, we obtain \( M = M_k \). This completes the proof.

3. We now consider the Hilbert space \( l^2(\mathbb{C}^k) \), \( k \geq 1 \), of norm-square-summable sequences of vectors of the \( k \)-dimensional unitary spaces \( \mathbb{C}^k \). Thus \( l^2(\mathbb{C}^k) \) consists of sequences

\[
x = \{x_m\}_{m=0}^{\infty}, \quad x_m \in \mathbb{C}^k,
\]

such that

\[
\sum_{m=0}^{\infty} \|x_m\|^2_* < \infty,
\]

where \( \|x_m\|_* \) is the norm of \( x_m \) in \( \mathbb{C}^k \), and

\[
\|x\| = \left( \sum_{m=0}^{\infty} \|x_m\|^2_* \right)^{1/2}.
\]

We shall say that a nonempty subset \( S \) of \( l^2(\mathbb{C}^k) \) is a cyclic set of an operator \( T \) on \( l^2(\mathbb{C}^k) \) if

\[
\bigvee_{n=0}^{\infty} \{T^n x: x \in S\} = l^2(\mathbb{C}^k).
\]

The following theorem generalises a result due to Nikolskii [4, Lemma 1].

**Theorem 4.** Let \( T \) be a unilateral weighted shift on \( l^2(\mathbb{C}^k) \) with weight sequence \( \{w_m\}_{m=0}^{\infty} \) such that \( \{w_m\}_{m=0}^{\infty} \) is of bounded variation and

\[
(5) \quad \delta = \sup_n \sum_{k=0}^{\infty} \left( \frac{w_{k+2} \cdots w_{k+n}}{w_2 \cdots w_n} \right)^2 < \infty.
\]

Then any set of \( k \)-vectors in \( l^2(\mathbb{C}^k) \), such that their first coordinates form a basis of \( \mathbb{C}^k \), is a cyclic set of \( T \).
Proof. Let \( x^{(i)} = \{x_m^{(i)}\}_{m=0}^{\infty}, i = 1, 2, \ldots, k \), be \( k \) elements of \( l^2(\mathbb{C}^k) \) such that \( \{x_0^{(i)}, x_0^{(2)}, \ldots, x_0^{(k)}\} \) is a basis in \( \mathbb{C}^k \). We assume without any loss of generality that \( \{x_0^{(i)}, x_0^{(2)}, \ldots, x_0^{(k)}\} \) is an orthonormal basis in \( \mathbb{C}^k \). Then

\[
T^n x^{(i)} = \left\{ 0, 0, \ldots, 0, w_{n-1}, \ldots, w_0 x_0^{(i)}, w_n \cdots w_1 x_1^{(i)}, \ldots \right\}.
\]

For each \( z \in \mathbb{C}^k \), let \( e_n(z) \) denote the element of \( l^2(\mathbb{C}^k) \) having \( z \) in the \( n \)th place and 0 elsewhere. Now observing that

\[
\sum_{m=0}^{\infty} \left( \frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \leq C w^2_n,
\]

we have

\[
\| \frac{T^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} - e_n(x^{(i)}) \|^2 = \sum_{m=0}^{\infty} \left( \frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \| x^{(i)}_{m+1} \|^2 \leq C w^2_n \| x^{(i)} \|^2.
\]

Since

\[
\{ e_n(x^{(i)}) \}_{n \geq 0, 1 \leq i \leq k}
\]

is an orthonormal basis in \( l^2(\mathbb{C}^k) \) and

\[
\sum_{n \geq 0, 1 \leq i \leq k} w^2_n \| x^{(i)} \|^2 < \infty,
\]

it follows that the system

\[
\left\{ \frac{T^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} \right\}_{n \geq 0, 1 \leq i \leq k}
\]

is a Riesz basis in \( l^2(\mathbb{C}^k) \), whence we conclude that \( \{x^{(i)}\}_{i=1}^{k} \) is a cyclic set of \( T \).

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