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SUBNORMALS IN C^*-ALGEBRAS

F. H. SZAFRANIEC

ABSTRACT. We prove, in a C^*-algebra set-up, the Bram improvement of Halmos' characterization of subnormals: (1) $\Rightarrow$ (2).

In what is the basic paper on subnormal operators [1], Bram proved that if S is a Hilbert space operator such that

$$\sum_{i,j=0}^{n} \left(S^{*i}S^{j}f_i, f_j\right) \geqslant 0, \qquad f_0,\ldots,f_n \in H,$$

then

$$(0) \qquad \sum_{i,j=0}^{n} \left(S^{*i+1}S^{j+1}f_i, f_j\right) \leqslant \|S\|^2 \sum_{i,j=0}^{n} \left(S^{*i}S^{j}f_i, f_j\right), \qquad f_0,\ldots,f_n \in H.$$

This simplifies Halmos' characterization of subnormal operators.

In his recent paper [2] Bunce refers to an element s in a unital C^*-algebra A as *subnormal* if

$$(1) \qquad \sum_{i,j=0}^{n} a_j^* s^{*i} s^j a_i \geqslant 0, \qquad a_0,\ldots,a_n \in A.$$

It is required for s to satisfy an inequality like (0) that

$$(2) \qquad \sum_{i,j=0}^{n} a_j^* s^{*i+1} s^{j+1} a_i \leqslant \|s\|^2 \sum_{i,j=0}^{n} a_j^* s^{*i} s^j a_i, \qquad a_0,\ldots,a_n \in A.$$

Then (p. 106) he writes "there should be a direct C^*-algebraic proof that (1) implies (2) without using Theorem 1 of [1] but we have been unable to find such a proof." Here we want to give such a proof adopting arguments of [3] and [4].

Take a positive linear functional φ on A and define $\omega(a, b) = \varphi(\sum_{i,j} b_j^* s^{*i} s^j a_i)$ where $a = \{a_i\}$, $b = \{b_i\}$ are finitely nonzero sequences of elements of A. Denote by X the linear space of such sequences. ω is a positive definite (hermitian) bilinear form on $X \times X$. Thus we have the Schwarz inequality $|\omega(a, b)|^2 \leqslant \omega(a, a)\omega(b, b)$.

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Denoting the element $\{0,0,\ldots,s^k a_0, s^k a_1,\ldots\}$ (where the first k coordinates are zero) of X by $\tilde{a}$, the Schwarz inequality gives us

$$\begin{aligned}\left(\varphi\left(\sum_{i,j=0} a_j^* s^{*i+k} s^{j+k} a_i\right)\right)^2 &= \left(\varphi\left(\sum_{j=0,i=k} a_j^* s^{*i} s^j\left(s^k a_{i-k}\right)\right)\right)^2\\ &= |\,\omega(\tilde{a},a)\,|^2 \leqslant \omega(a,a)\omega(\tilde{a},\tilde{a})\\ &= \varphi\left(\sum_{i,j=0} a_j^* s^{*i} s^j a_i\right)\varphi\left(\sum_{i,j=0} a_j^* s^{*i+2k} s^{j+2k} a_i\right).\end{aligned}$$

Using this and a mathematical induction we have

$$\left(\varphi\left(\sum_{i,j} a_j^* s^{*i+1} s^{j+1} a_i\right)\right)^{2^k} \leqslant \left(\varphi\left(\sum_{i,j} a_j^* s^{*i} s^j a_i\right)\right)^{2^k-1} \cdot \varphi\left(\sum_{i,j} a_j^* s^{*i+2^k} a^{j+2^k} a_i\right)$$

and consequently

$$\left(\varphi\left(\sum_{i,j} a_j^* s^{*i+1} s^{j+1} a_i\right)\right)^{2^k} \leqslant \left(\varphi\left(\sum_{i,j} a_j^* s^{*i} s^j a_i\right)\right)^{2^k-1} \|\varphi\|\left(\sum_i \|a_i\|\,\|s^i\|\right)^2 \|s\|^{2^{k+1}}.$$

Taking the 2^k th root and passing to the limit we get

$$\varphi\left(\sum_{i,j=0}^{n} a_j^* s^{*i+1} s^{j+1} a_i\right) \leqslant \|s\|^2 \varphi\left(\sum_{i,j=0}^{n} a_j^* s^{*i} s^j a_i\right).$$

Since φ is arbitrary, in conclusion, we establish (2).

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UNIWERSYTET JAGIELLOŃSKI, INSTYTUT MATEMATYKI, UL. REYMONTA 4, 30059 KRAKÓW, POLAND