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MULTIPLIERS AND ASYMPTOTIC BEHAVIOUR OF THE FOURIER ALGEBRA OF NONAMENABLE GROUPS

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ABSTRACT. Let G be a locally compact group and $A(G)$ the algebra of matrix coefficients of the regular representation. We prove that G is amenable if and only if there exist functions $u \in A(G)$ which vanish at infinity at any arbitrarily slow rate. The "only if" part of the result was essentially known. With the additional hypothesis that G be discrete, we deduce that G is amenable if and only if every multiplier of the algebra $A(G)$ is a linear combination of positive definite functions. Again, the "only if" part of this result was known.

1. Introduction. Let G be a locally compact group; let $C_0(G)$ be the algebra of continuous complex-valued functions on G which vanish at infinity. We use the definitions and the terminology of [3]. We let $B(G)$ be the Fourier-Stieltjes algebra consisting of all matrix coefficients of unitary representations of G, and $A(G)$ the Fourier algebra consisting of matrix coefficients of the regular representation. Regarded as the dual space of $C^*(G)$ (completion of $L^1(G)$ in the minimal regular norm), $B(G)$ is a Banach algebra under pointwise multiplication and $A(G)$ is a closed ideal in $B(G)$. We let $VN(G)$ be the von Neumann algebra of the regular representation λ of G, which is the dual space of $A(G)$. It is known and easy to prove that if G is amenable there exist functions in $A(G)$ which vanish at infinity at arbitrarily slow rates. More specifically, if G is amenable, for every $f \in C_0(G)$, there exist $u \in A(G)$ and $g \in C_0(G)$ such that $f(x) = u(x)g(x)$. This means of course that $f(x) = o(u(x))$ as $x \to \infty$. We show that this property is characteristic of amenable groups. Theorem 1 contains this result and other equivalent characterizations of amenability.

Hence if G is nonamenable, the coefficients of the regular representation must satisfy some condition of decrease at infinity. For particular nonamenable groups specific significant conditions are known; for instance, if G is a semisimple Lie group with finite center, the Kunze-Stein phenomenon [2] implies that $A(G) \subset \bigcap_{p>2} L^p(G)$, and if G is a free group with at least two generators, a coefficient $u(x)$ of the regular representation must satisfy the condition $\{\sum_{|x|=n} |u(x)|^2\}^{1/2} = O(n)$ where $|x|$ denotes the length of the reduced word x [9, Theorem 3.1, p. 291].

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Our result shows that these phenomena, which we could call "generalized Kunze-Stein phenomena," are characteristic of nonamenable groups. For the case of a discrete nonamenable group G, we use the result above to prove that there exist multipliers of $A(G)$ which are not in $B(G)$. This fact was known for groups containing a free group with two generators [7].

2. The general case. Recall that there is a natural module action of $A(G)$ on $VN(G)$ defined by

$$\langle u \cdot T, v \rangle = \langle T, uv \rangle,$$

for u, v in $A(G)$ and T in $VN(G)$. This allows us to introduce the following definition.

DEFINITION 1. A bounded linear operator Φ of $C_0(G)$ into $VN(G)$ is called a multiplier of $C_0(G)$ into $VN(G)$ if, for every f in $C_0(G)$ and every u in $A(G)$,

$$\Phi(uf) = u \cdot \Phi(f).$$

Let $\mathfrak{M}(C_0, VN)$ be the space of multipliers of $C_0(G)$ into $VN(G)$. It is not difficult to see that $\mathfrak{M}(C_0, VN)$ can be identified with a subspace of $VN(G)$. It consists precisely of those elements T in $VN(G)$ such that $|||T||| < \infty$ where

$$|||T||| = \sup\{\|u \cdot T\|_{VN}: u \in A(G), \|u\|_\infty \leq 1\}.$$

The norm $|||\cdot|||$ is exactly the operator norm of $\mathfrak{M}(C_0, VN)$.

Let $M(G)$ be the space of bounded regular measures; it is clear that, for every μ in $M(G)$ and every u in $A(G)$, $u\mu \in M(G)$ where $d(u\mu) = u\,d\mu$.

DEFINITION 2. A bounded linear operator Γ of $A(G)$ into $M(G)$ is called a multiplier of $A(G)$ into $M(G)$ if, for every u and v in $A(G)$,

$$\Gamma(uv) = u\Gamma(v).$$

Let $\mathfrak{M}(A, M)$ be the Banach space of multipliers of $A(G)$ into $M(G)$ with the operator norm. Let $\mathfrak{M}(A, L^1)$ be the subspace of $\mathfrak{M}(A, M)$ which consists of those elements Γ such that $\Gamma(u) \in L^1(G)$ for every u in $A(G)$.

It is clear that for every f in $L^1(G)$ the operator $\Gamma(u) = uf$ is a multiplier of $A(G)$ into $L^1(G)$.

It is easy to see that the space $\mathfrak{M}(A, M)$ is isometrically isomorphic to the space $\mathfrak{M}(C_0, VN)$, since $M(G)$ and $VN(G)$ are, respectively, the dual spaces of $C_0(G)$ and $A(G)$.

The isomorphism is given by the following formula:

$$\langle \Phi(f), v \rangle = \langle f, \Gamma(v) \rangle$$

for Γ in $\mathfrak{M}(A, M)$, Φ in $\mathfrak{M}(C_0, VN)$, f in $C_0(G)$ and v in $A(G)$.

If $\mu \in M(G)$ then $\Gamma(u) = u\mu$, for u in $A(G)$, is a multiplier of $A(G)$ into $M(G)$ and $|||\Gamma||| \leq \|\mu\|$.

Moreover every multiplier is a Radon measure, generally unbounded. Indeed suppose that K is a compact subset of G and let v_K be a function in $A(G)$ such that $v_K(x) = 1$ for every x in K. If u is a function in $A(G)$ with support in K, then

$$\begin{aligned} |\langle T, u \rangle| &= |\langle T, uv_K \rangle| = |\langle u \cdot T, v_K \rangle| \\ &= |\langle u, \Gamma(v_K) \rangle| \leq \|u\|_\infty\, |||\Gamma|||\, \|v_K\|_A \end{aligned}$$

where T is the element in $\mathfrak{M}(C_0, VN)$ corresponding to Γ in $\mathfrak{M}(A, M)$.

The Radon measures, generally, are unbounded, but we shall see that every multiplier is a bounded Radon measure if and only if G is amenable.

For every multiplier Γ in $\mathfrak{M}(A, M)$ let $|\Gamma|$ be the absolute value of Γ in the measure theoretic sense. Then also $|\Gamma|$ is a multiplier of $A(G)$ into $M(G)$ and $|||\Gamma||| = |||\,|\Gamma|\,|||$.

This follows from the fact that, for every u in $A(G)$ with compact support K,

$$\langle u, |\Gamma|(v_K)\rangle = \langle u, |\Gamma(v_K)|\rangle \quad \text{and} \quad \|\Gamma(u)\| = \|\,|\Gamma|(u)\,\|.$$

Finally, we observe that if S and T are Radon measures such that $|S| \leq |T|$, then $|||\,|S|\,||| \leq |||\,|T|\,|||$.

Hence if T is a multiplier then also S is a multiplier because the multipliers of $A(G)$ into $M(G)$ are exactly the Radon measures T such that $|||T||| < \infty$.

Definition 3. Let X be the subspace of $C_0(G)$ defined as follows:

$$X = \left\{ h\colon h(x) = \sum_{i=1}^{\infty} u_i(x)g_i(x),\, u_i \in A(G),\, g_i \in C_0(G),\, \sum_{i=1}^{\infty} \|u_i\|_A \|g_i\|_\infty < \infty \right\}.$$

We define the norm:

$$\|h\|_X = \inf\left\{ \sum_{i=1}^{\infty} \|u_i\|_A \|g_i\|_\infty \colon \sum_{i=1}^{\infty} u_i(x)g_i(x) = h(x) \right\}.$$

It is not difficult to show that, with the norm $\|\cdot\|_X$, the space X is a Banach space. We can also prove the following lemma.

Lemma 1. *The space $\mathfrak{M}(C_0, VN)$ is isometrically isomorphic to the dual space of X; the duality is given by the following formula*:

$$\langle \Phi, h\rangle = \sum_{i=1}^{\infty} \langle \Phi(g_i), u_i\rangle$$

for h in X, $h = \sum_i u_i g_i$, and Φ in $\mathfrak{M}(C_0, VN)$.

Proof. We can apply the proof of [6, Theorem 1], observing that $C_0(G)$ always has an approximate identity, bounded in the supremum norm, which consists of elements of $A(G)$.

Theorem 1. *If G is a locally compact group, the following are equivalent*:

(a) *G is amenable.*

(b) *For every f in $C_0(G)$ there exist u in $A(G)$ and g in $C_0(G)$ such that $u(x)g(x) = f(x)$.*

(c) $\mathfrak{M}(A, M) \simeq \mathfrak{M}(C_0, VN) = M(G)$.

(d) $\mathfrak{M}(A, L^1) = L^1(G)$.

Proof. (a) implies (b). If G is amenable there is a bounded approximate identity in $A(G)$ [13]. Since $C_0(G)$ is an $A(G)$-module, (b) follows from the factorization theorem for Banach modules [11, (32.22)]. If (b) holds, the space X defined above is $C_0(G)$ and (c) follows from Lemma 1.

If (c) holds for every Γ in $\mathfrak{M}(A, L^1)$ there exists a bounded measure μ in $M(G)$ such that $\Gamma(u) = u\mu$ and $u\mu \in L^1(G)$ for every u in $A(G)$; hence, $\mu \in L^1(G)$ and (d) follows. We prove now that (d) implies (a).

It is known [**14**] that G is amenable if and only if $\|f\|_1 = \|\lambda(f)\|$ for every nonnegative f in $L^1(G)$. We observe that, for f in $L^1(G)$ and u in $A(G)$,

$$\|u \cdot \lambda(f)\| = \|\lambda(uf)\| \leq \|u\|_\infty \|\lambda(|f|)\|.$$

Indeed if $g \in L^2(G)$,

$$\begin{aligned}\|\lambda(uf)g\|_2 &= \|uf * g\|_2 \leq \|\, |uf| * |g|\, \|_2 \\ &\leq \|\, \|u\|_\infty |f| * |g|\, \|_2 \leq \|u\|_\infty \|\lambda(|f|)\| \, \|g\|_2.\end{aligned}$$

This means that if $f \geq 0$, then $\|u \cdot \lambda(f)\| \leq \|u\|_\infty \|\lambda(f)\|$ and therefore $|||\lambda(f)||| \leq \|\lambda(f)\|$ for a nonnegative f in $L^1(G)$.

If (d) holds, the norm $\|\cdot\|_1$ of $L^1(G)$ is equivalent to $|||\cdot|||$ (by the closed graph theorem). In particular, $\|f\|_1 \leq k|||\lambda(f)|||$ for every f in $L^1(G)$. Hence $\|f\|_1 \leq k\|\lambda(f)\|$ for every nonnegative f in $L^1(G)$. But if $f \geq 0$,

$$\|f * f^*\|_1 = \|f\|_1^2.$$

This implies that $\|f\|_1 \leq \sqrt{k}\, \|\lambda(f)\|$, which means $\|f\|_1 \leq \|\lambda(f)\|$ and (a) follows from [**14**, Theorem 1].

In the following corollary we prove that, for nonamenable groups, the condition of decrease at infinity of the Fourier algebra is a summability condition with respect to a positive measure.

COROLLARY. *If G is nonamenable there exists a positive Radon measure μ such that*:

(a) $\mu(G) = +\infty$,

(b) $A(G) \subset L^p(\mu)$ *for every* $1 \leq p \leq \infty$.

PROOF. If G is nonamenable there exists a multiplier Γ of $A(G)$ into $L^1(G)$ which is not in $L^1(G)$. Then also $|\Gamma|$ is a multiplier of $A(G)$ into $L^1(G)$ which is not in $L^1(G)$. So $\mu = |\Gamma|$ is a positive Radon measure such that

$$\mu(G) = +\infty \quad \text{and} \quad A(G) \subset L^1(\mu).$$

Finally $A(G) \subset L^\infty(\mu)$, obviously, and (b) follows.

REMARKS. (1) Observe that the statement (d) of Theorem 1 is true for every $1 < p < \infty$. Indeed let $\mathfrak{M}(A, L^p)$ be the space of multipliers of $A(G)$ into $L^p(G)$ for $1 < p < \infty$, i.e. the Banach space, with the operator norm, of the bounded linear operators Φ of $A(G)$ into $L^p(G)$, such that $\Phi(uv) = u\Phi(v)$ for every u and v in $A(G)$. So, if G is amenable there is a bounded approximate identity in $A(G)$; hence, $\mathfrak{M}(A, L^p) = L^p(G)$ for every $1 < p < \infty$. Conversely, if G is nonamenable and $1 < p < \infty$, then there exists a multiplier Γ of $A(G)$ into $L^1(G)$ which is not in $L^1(G)$. But if K is a compact subset of G then we can find a function f such that $\Gamma(u) = fu$ for every u in $A(G)$ whose support is in K; hence, $\Phi(u) = |f|^{1/p}u$ is a multiplier of $A(G)$ into $L^p(G)$ which is not in $L^p(G)$.

(2) The space of multipliers of $A(G)$ into $l^1(G)$ for discrete groups was studied by M. Bożejko in [**1**]; he proves that a discrete group G is amenable if and only if $l^1(G) = l^1[G]$ where $l^1[G]$ is the closure, in $\mathfrak{M}(A, l^1)$, of the space of functions with finite support.

(3) Observe that, in the corollary, the measure μ is 2-admissible since $A(G) \subset L^1(\mu)$ (see [**14, 8, 5**]).

3. The discrete case. Let G be a discrete group; then $M(G) = l^1(G)$ and $\mathfrak{M}(A, l^1)$, defined above, is exactly the space of all functions φ on G such that $\varphi u \in l^1(G)$ for every u in $A(G)$.

DEFINITION 4. If X and Y are two spaces of functions on G, let $\mathfrak{M}(X, Y)$ be the space of the multipliers of X into Y, i.e. the functions φ on G such that $\varphi g \in Y$ for every g in X.

If X and Y are Banach spaces, $\mathfrak{M}(X, Y)$ is a Banach space in the operator norm.

LEMMA 2. $\mathfrak{M}(l^\infty, B) = l^2(G)$.

PROOF. Let φ be in $\mathfrak{M}(l^\infty, B)$ and let F be a finite subset of G. Let $\varphi_F = \chi_F \varphi$, where χ_F is the characteristic function of the set F; then

$$\|\varphi_F\|_A = \|\varphi\chi_F\|_B \leq |||\varphi|||\, \|\chi_F\|_\infty = |||\varphi|||$$

where $||| \cdot |||$ is the norm of $\mathfrak{M}(l^\infty, B)$.

Define as in [**15**, Definition 4] the space L^{2s}, $s = 1, 2, \ldots,$ as the completion of the space of the finitely supported functions with the norm $\|f\|_{L^{2s}} = ([f * f^*]^s(1_G))^{1/2s}$, where the power is meant as a convolution power. Then the space L^2 is the same as l^2, and the proof of [**15**, Theorem 2] shows that there exists a function u of modulus one on F such that

$$\|u\varphi_F\|_{L^{2s}} \leq 2\sqrt{s}\, \|\varphi_F\|_{L^2},$$

[**15**, p. 503]. In particular $\|u\varphi_F\|_{L^4} \leq 2\sqrt{2}\, \|\varphi_F\|_{l^2}$. But

$$\|f\|_{L^2} \leq \|f\|_A^{1/3} \|f\|_{L^4}^{2/3}.$$

(This inequality can be deduced from the corresponding inequality for ordinary L^p-spaces, where L^1 takes the place of $A(G)$, by using the fact that the trace $\mathrm{tr}(f) = f(1_G)$ acts as an ordinary integral on any commutative $*$-subalgebra of $VN(G)$ [**12**; **15**, Remark 4].)

We conclude that for some u of modulus one,

$$\|\varphi_F\|_{l^2} = \|u\varphi_F\|_{l^2} \leq \|u\varphi_F\|_A^{1/3} \|u\varphi_F\|_{L^4}^{2/3}$$

$$\leq (2\sqrt{2}\,)^{2/3} \|u\varphi_F\|_A^{1/3} \|\varphi_F\|_{l^2}^{2/3};$$

hence $\|\varphi_F\|_{l^2} \leq 8\|u\varphi_F\|_A = 8\|u\varphi_F\|_B \leq 8|||\varphi|||$.

Therefore $\varphi \in l^2(G)$, as F is arbitrary.

THEOREM 2. *If G is a discrete group, the following are equivalent*:

(a) *G is amenable.*

(b) $\mathfrak{M}(A, A) = B(G)$.

PROOF. It is known that (a) implies (b) for locally compact groups [**5, 10, 16**]. We prove now that (b) implies (a).

If G is nonamenable there exists a multiplier f of $A(G)$ into $l^1(G)$ which is not in $l^1(G)$; it follows from Theorem 1.

Then $h = |f|^{1/2}$ is a multiplier of $A(G)$ into $l^2(G)$ which is not in $l^2(G)$, as is easily seen. From Lemma 2 it follows that there exists a function φ in $l^\infty(G)$ such that φh is not in $B(G)$, but $\varphi h \in \mathfrak{M}(A, l^2) \subset \mathfrak{M}(A, A)$. Hence $B(G) \neq \mathfrak{M}(A, A)$.

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