THE NONHOMOGENEITY OF THE $E$-TREE—ANSWER TO A PROBLEM RAISED BY D. JENSEN AND A. EHRENFEUCHT

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Abstract. We prove that the ordered system of all $C^1EP$'s, under the order "admits embedding in" is not homogeneous. This answers a problem raised in [2].

1. Introduction. We assume familiarity with [2]. $L$ denotes the lattice of all $\forall_1$-sentences of Peano Arithmetic (PA) modulo PA. $\alpha, \beta, \gamma, \ldots$ denote elements of $L$ (we often identify $\forall_1$-sentences with their equivalence classes). $0$ and $1$ denote respectively the minimum element and the maximum element of $L$.

By the $E$-tree we mean, the class of all prime filters of $L$ under the partial ordering of reverse inclusion $\subseteq$. By a $C^1EP$ is meant the set of all existential sentences (without parameters) satisfied in some model of PA. The following results are well known (see [2] and [4]).

Lemma 1.1. $F$ is a prime filter of $L$ iff

$$-(L \setminus F)$$ is a $C^1EP$.

This gives an isomorphism between the $E$-tree and the ordered system of all $C^1EP$'s.

Lemma 1.2. (i) The set of the predecessors of an element of the $E$-tree is totally ordered.

(ii) The $E$-tree has a minimum element (i.e. $L \setminus \{0\}$) and each of its branches has a maximal element.

Jensen and Ehrenfeucht ask [2, p. 243] whether the $E$-tree is homogeneous in the sense that any pair of nonminimal, nonmaximal elements can be exchanged by an automorphism.

2. Preliminary results.

Lemma 2.1. The $E$-tree has an element $F$ such that

(i) $F$ is not maximal,

(ii) $F$ is not minimal,

(iii) $F$ has no immediate predecessor,

(iv) if $B$ is any branch of the $E$-tree containing $F$, then $F$ has an immediate successor in $B$.

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Proof. Let \( \theta \) be a \( \forall_1 \)-sentence independent of PA such that \( \text{PA} + \neg \theta \) and \( \text{PA} \) have the same \( \forall_1 \)-theorems; (such a formula exists by a result of Kreisel, cf. §1 of [1]). We denote by \( E_\theta \) the class of all prime filters of \( L \) containing \( \theta \) and ordered by \( \supset \). It is easily shown that each branch of \( E_\theta \) has a maximum element. Therefore \( E_\theta \) has (at least) one maximal element \( F_\theta \). We will show that \( F_\theta \) has the required properties.

(i) Let \( I = L \setminus F_\theta \), we have \( \theta \notin I \). Denote by \( T \) the theory \( \text{PA} + \neg I + \neg \theta \). \( T \) is consistent because if \( \text{PA} + \neg I + \theta \), then

\[
\exists \varphi \in I \quad \text{PA} + \neg \varphi + \theta,
\]

and thus

\[
\text{PA} + \varphi.
\]

This is impossible because \( \text{PA} + \neg \varphi \) is consistent. Obviously, the prime filter of all \( \forall_1 \)-sentences true in any model of \( T \) is properly contained in \( F_\theta \). So \( F_\theta \) is not maximal.

(ii) If \( F_\theta = L \setminus \{0\} \), then \( E_\theta = \{L \setminus \{0\}\} \) and the only prime filter of \( L \) containing \( \theta \) is \( \forall_1(N) \). We infer that \( \text{PA} + \theta \equiv \text{PA} + (L \setminus \{0\}) \). This is impossible because \( \text{PA} + \theta \) is an R.E. theory and \( \text{PA} + (L \setminus \{0\}) \) is a \( \pi^0_1 \)-non-R.E. theory.

(iii) Suppose that \( F_\theta \) has an immediate predecessor \( F' \). Then

\[
\forall \beta \in L \setminus F_\theta \quad \text{PA} + F_\theta + \beta + F' \]

(because, if this theory is consistent, the class of all \( \forall_1 \)-sentences true in any model of \( T \) is a prime filter of \( L \) containing properly \( F_\theta \) and therefore \( F' \); if \( \text{PA} + F_\theta + \beta \) is not consistent, the result is obvious). Therefore

\[
\forall \alpha \in F' \setminus F_\theta \quad \forall \beta \in L \setminus F_\theta \quad \exists \gamma \in F_\theta \quad \text{PA} + \gamma + \beta + \alpha,
\]

or

\[
(*) \quad \text{PA} + \neg \alpha + \beta + \neg \gamma.
\]

We also have that \( F_\theta \) is a maximal element of \( E_\theta \) and therefore \( \forall \xi \in F_\theta \quad \text{PA} + (L \setminus F_\theta) + \theta + \neg \xi \) is an inconsistent theory (because, if this theory is consistent, the prime filter of all \( \forall_1 \)-sentences true in any model of this theory is an element of \( E_\theta \) properly contained in \( F_\theta \)). Therefore,

\[
(**) \quad \forall \xi \in F_\theta \quad \exists \rho \in L \setminus F_\theta \quad \text{PA} + \theta + \xi \lor \rho.
\]

Let \( \alpha \in F' \setminus F_\theta \). By a result of Solovary (cf. [1, Theorem 2.7]), we know that there is a \( \forall_1 \)-sentence \( \varphi \), independent of \( \text{PA} + \theta + \neg \alpha \), such that

(I) \( \text{PA} + \theta + \neg \alpha + \varphi \) and \( \text{PA} + \theta + \neg \alpha \) have the same \( \exists_1 \)-theorems,

(II) \( \text{PA} + \theta + \neg \alpha + \neg \varphi \) and \( \text{PA} + \theta + \neg \alpha \) have the same \( \forall_1 \)-theorems. \( \varphi \notin F_\theta \);

because, if \( \varphi \in F_\theta \) then, by (**) we have

\[
\exists \varphi' \in L \setminus F_\theta \quad \text{PA} + \theta + \varphi \lor \varphi',
\]

\[
\text{PA} + \theta + \neg \alpha + \varphi \lor \varphi',
\]

\[
\text{PA} + \theta + \neg \alpha + \neg \varphi + \varphi',
\]

\[
\text{PA} + \theta + \neg \alpha + \varphi',
\]
and
\[ PA \vdash \theta \Rightarrow \alpha \lor \varphi', \]
but \( \theta \in \mathcal{F}_{\alpha}, \alpha \lor \varphi' \notin \mathcal{F}_{\alpha} \) and \( \theta \leq \alpha \lor \varphi' \). Contradiction!

If \( \varphi \in L \setminus \mathcal{F}_{\alpha} \), we have by (\( \ast \))
\[ \exists \gamma \in \mathcal{F}_{\alpha} \quad PA + \neg \alpha + \varphi \vdash \neg \gamma, \]
\[ PA + \theta + \neg \alpha + \varphi \vdash \neg \gamma, \]
\[ PA + \theta + \neg \alpha \vdash \neg \gamma, \]
and
\[ \exists \gamma \in \mathcal{F}_{\alpha} \quad PA + \theta \land \gamma \Rightarrow \alpha, \]
but \( \alpha \notin \mathcal{F}_{\alpha}, \theta \land \gamma \in \mathcal{F}_{\alpha} \) and \( \theta \land \gamma \leq \alpha \). Contradiction! \( \mathcal{F}_{\alpha} \) has therefore no immediate predecessor.

(iv) Let \( B \) be a branch of the \( \mathcal{E} \)-tree containing \( \mathcal{F}_{\alpha} \). \( \mathcal{F}_{\alpha} \) is a maximal element of \( \mathcal{E}_{\alpha} \), and therefore \( \mathcal{F}_{\alpha} \) is the greatest element of \( B \) containing \( \theta \). Let \( A = \{ F \in B \mid \theta \notin F \} \).

It is straightforward to check that \( F' = \bigcup_{F \in A} F \) is the lowest element of \( B \) which does not contain \( \theta \). \( F' \) is, of course, an immediate successor of \( \mathcal{F}_{\alpha} \).

**Lemma 2.2.** If \( F \) is any maximal element of the \( \mathcal{E} \)-tree, then \( F \) has no immediate predecessor.

**Proof.** We use the same kind of argument as in the proof of Lemma 2.1(iii). Let \( \theta = 1 \). (We delete, of course, the requirement “\( PA + \neg \theta \) and \( PA \) have the same \( \forall \_\theta \)-theorems” which is not used in the proof of Lemma 2.1(iii).) Now \( \mathcal{F}_{\alpha} \) becomes any maximal element of the \( \mathcal{E} \)-tree.

3. The main result.

**Theorem 3.1.** The \( \mathcal{E} \)-tree is not homogeneous.

**Proof.** This is an immediate consequence of Lemmas 2.1 and 2.2, for, in the notation of Lemma 2.1, \( \mathcal{F}_{\alpha} \) and its immediate successor \( F' \) are neither minimal nor maximal and yet cannot be exchanged by an automorphism of the \( \mathcal{E} \)-tree.

**References**


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