PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, April 1982

INNER POINTS AND BREADTH IN CERTAIN COMPACT SEMILATTICES

D. R. BROWN[1] AND J. W. STEPP

ABSTRACT. A point $x \in X$ is inner if there exists an open set U containing x such that for each open set V with $x \in V \subseteq U$, the inclusion homomorphism i^*: $H^*(X, X \setminus V) \to H^*(X, X \setminus U)$ is nontrivial. In this note it is proved that, if X is a compact, chainwise connected topological semilattice of codimension n, and x is a point of breadth $n+1$, then x is an inner point.

1. Introduction. A topological semilattice is a commutative, idempotent topological semigroup; equivalently, it is a partially ordered Hausdorff space in which each pair of elements have a greatest lower bound, and the function $(a, b) \to \operatorname{glb}\{a, b\}$ is jointly continuous. These objects have been investigated extensively in the past decade, particularly with regard to their relationship to the continuous lattices of D. Scott [**5**]. In this paper, all semilattices to be discussed are defined on continua, and are additionally assumed to be chainwise connected [**8**]; that is, if $x < y$, then there exists a subcontinuum A which is simultaneouly a topological arc with endpoints x and y and a totally ordered subset in the order inherited from the space. This subclass, which includes all compact, connected lattices, has received considerable attention over the last several years [**1, 3, 8, 9**].

In any semilattice S, if $x \in S$, then $L(x) = \{y\colon y \leqslant x\}$, $M(x) = \{y\colon y \geqslant x\}$. Since the equivalence mentioned in the first sentence of the paper is established by defining $x \leqslant y$ if $xy = x$, it is immediate that $L(x) = xS$, the principal ideal generated by x. No such algebraic alternative exists for $M(x)$; we shall nevertheless refer to this set as the principal filter generated by x. By a well-known theorem of R. J. Koch [7] on the existence of arcs in certain partially ordered spaces, it follows that, if X is a continuum and $M(x)$ is connected for each $x \in S$, then S is chainwise connected, as are every $L(x)$ and $M(x)$. Thus S is chainwise connected if and only if $M(x)$ is connected for each $x \in S$, and we shall use these ideas interchangeably.

We shall invoke the cohomology theory of Alexander-Spanier-Wallace, as set forth, for example, in [**13**]. For compact Hausdorff spaces, this theory satisfies the continuity axiom and is thus equivalent to Čech cohomology. In particular, we will need the exactness of compact pairs, the Mayer-Vietoris Theorem, and the Strong

---

Received by the editors June 5, 1981 and, in revised form, August 3, 1981; presented to the Society, January 1982, in Cincinatti, Ohio.

1980 *Mathematics Subject Classification*. Primary 22A26; Secondary 54H12.

Key words and phrases. Inner points, codimension.

[1] Research partially supported by UH Grant REG-1980.

Excision Theorem which establishes isomorphism between the groups of the (compact) pair $(A \cup B, A)$ and those of $(B, A \cap B)$. The Vietoris Mapping Theorem will be used in the following weak form: if $f\colon X \to Y$ is a closed map such that $H^p(f^{-1}(y)) = 0$ for all $p \geqslant 0$ (where H^0 is assumed reduced), then f^* is an isomorphism in all dimensions. For a proof of this theorem in ASW cohomology and in much greater generality, see **[10]**. Our codimension function is that of Haskell Cohen **[4]**: $\mathrm{cd}(X) \leqslant n$ provided $i^*\colon H^n(X) \to H^n(A)$ is onto for every closed subset A of X. A nonempty space is acyclic provided it has trivial reduced cohomology; that is, all of its groups are trivial. Thus, an acyclic space has codimension $\leqslant n$ if its closed subsets all have trivial n-cohomology. Finally, we shall have occasion to use the following corollary of Wallace's Acyclicity Theorem **[15]**: a compact, connected topological semilattice is acyclic.

The breadth of a semilattice S, denoted $\mathrm{Br}(S)$, is the smallest positive integer n such that any finite subset F of S has a subset F_1 of at most n elements such that $\inf(F_1) = \inf(F)$. Semilattices of finite breadth have received periodic attention in the literature **[2, 8, 9]**. Lawson **[8**, p. 210; **9**, p. 557] proved these inequalities for a large class of topological semilattices which includes the compact, chainwise connected ones: $\mathrm{cd}(S) \leqslant \mathrm{Br}(S) \leqslant \mathrm{cd}(S) + 1$, where the first relation becomes equality if S has an identity. Thus a compact, connected lattice L has $\mathrm{cd}(L) = \mathrm{Br}(L)$, but there exist chainwise connected semilattices of codimension n and breadth $n + 1$. Indeed, if $I = [0, 1]$ with the usual order, then $S = \{(x, y) \in I \times I\colon xy = 0\}$ is an example for $n = 1$. A more motivating example, where $n = 2$, is given in [**3**, p. 41].

The idea of an intrinsic boundary for topological spaces has existed for quite a long time, particularly in topological algebra. An informal survey of the history of this subject up until 1965 occurs in **[6]**; subsequently, Lawson and Madison **[11, 12]** weakened previous definitions to the following. A point $x \in X$ is peripheral if, for each open set U containing x, there exists an open set V containing x, $V \subseteq U$, such that the inclusion homomorphism $i^*\colon H^p(X, X\setminus V) \to H^p(X, X\setminus U)$ is trivial in all dimensions. Points which are not peripheral are called inner points. This note proves the following.

THEOREM. *Let S be a compact, chainwise connected topological semilattice of codimension n. If* $\mathrm{Br}(S) = n + 1$, *then any point of breadth $n + 1$ is an inner point.*

Points of breadth $n + 1$ are, of course, those points possessing an irreducible representation as a product of $n + 1$ elements. The uniqueness of this result lies in the following observation: previously, theorems involving algebraic properties of elements and their consequent positions have concluded such elements to be peripheral. For example, maximal elements in compact semilattices are peripheral and the identity of a clan is peripheral **[12]**. To the authors' knowledge, the theorem above is the first one in which algebraic conditions on an element force it to be an inner point.

2. Technical considerations.

LEMMA 1. *Let X be an acyclic space. Then $x \in X$ is an inner point of X if and only if there exists an open set U containing x such that for each open set V containing x, $V \subseteq U$, the homomorphism induced by the inclusion map j: $X \setminus U \to X \setminus V$ is nonzero in some dimension.*

PROOF. By the acyclicity of X and the exactness of the pair sequences $(X, X \setminus U)$ and $(X, X \setminus V)$, the horizontal arrows in the diagram below are isomorphisms in all dimensions:

$$\begin{array}{ccc} H^p(X \setminus V) & \to & H^{p+1}(X, X \setminus V) \\ j^* \downarrow & & i^* \downarrow \\ H^p(X \setminus V) & \to & H^{p+1}(X, X \setminus U) \end{array}$$

The conclusion follows from independence of path and the observation that x is an inner point if and only if i^* is nonzero in some dimension.

LEMMA 2. *Let A be a closed subset of X, a compact Hausdorff space. Let* $\mathrm{cd}(X) = n$, $H^n(X) = 0 = H^{n-1}(A)$. *If x is an inner point in dimension $n - 1$ relative to A, then x is an inner point relative to X (in the same dimension).*

PROOF. By the previous lemma, there exists a set U, open in A and containing x, such that j^*: $H^{n-1}(A \setminus V) \to H^{n-1}(A \setminus U)$ is not trivial for every A-open set $V \subseteq U$ such that $x \in V$.

Let U_1 be an open set in X such that $U_1 \cap A = U$, and let V_1 be any open set in X such that $x \in V_1 \subseteq U_1$. Consider the Mayer-Vietoris exact sequence:

$$H^{n-1}(X \setminus V_1) \times H^{n-1}(A) \to H^{n-1}((X \setminus V_1) \cap A) \to H^n(A \cup (X \setminus V_1)).$$

Since $\mathrm{cd}(X) = n$ and $H^n(X) = 0$, the final group in the above triple is zero. Because $H^{n-1}(A) = 0$, it follows that i_1^*, the homomorphism induced by the inclusion of $A \setminus V_1$ into $X \setminus V_1$, is onto $H^{n-1}(A \setminus V_1)$. Since $A \cap V_1$ is an A-open set contained in U and containing x, j^* is not zero. Finally, from the diagram

$$\begin{array}{ccc} H^{n-1}(X \setminus V_1) & \xrightarrow{j_1^*} & H^{n-1}(X \setminus U_1) \\ i_1^* \downarrow & & i_U^* \downarrow \\ H^{n-1}(A \setminus V_1) & \xrightarrow{j^*} & H^{n-1}(A \setminus U), \end{array}$$

we have $0 \neq j^* i_1^* = i_U^* j_1^*$, whence $j_1^* \neq 0$, which shows that x is an inner point relative to X.

LEMMA 3. *Let $\{A_i\}_{i=1}^n$ be a collection of acyclic sets such that $\bigcap_{i \in I} A_i$ is acyclic for every nonempty $I \subseteq \{1, \ldots, n\}$. Then $\bigcup \{A_i: i = 1, \ldots, n\}$ is acyclic.*

PROOF. Each A_i is connected, as are pairwise intersections; this proves the lemma in dimension 0, for all n. If $n = 2$, the result is immediate by Mayer-Vietoris in all dimensions. If $n > 2$, the lemma follows by direct application of mathematical induction.

LEMMA 4. *Let S be a compact, chainwise connected semilattice with* $\mathrm{cd}(S) = n$. *Let z be an element of breadth $n+1$, with irreducible representation $z = x_1x_2\cdots x_{n+1}$. Let $y_i = x_1x_2\cdots\hat{x}_i\cdots x_{n+1}$, where "$\hat{x}$" means "delete x". Then, for any proper subset $I \subseteq \{1,\ldots,n+1\}$, $\bigcap\{M(y_i): i \in I\} \neq \square$, whereas*

$$\bigcap\{M(y_i): i = 1,\ldots,n+1\} = \square.$$

PROOF. Choose $j \in \{1,\ldots,n+1\}\setminus I$. Then $x_j \geqslant y_i$ for every $i \in I$. Hence $\bigcap\{M(y_i): i \in I\} \neq \square$.

Next, suppose there exists $a \in \bigcap\{M(y_i): i = 1,\ldots,n+1\}$. Let $b_i = ax_i$, $i = 1,\ldots,n+1$. If $b_i = b_j$, $i \neq j$, then $ax_ix_j = ax_j$, whence $z = az = ax_1\cdots x_{n+1} = ay_i = y_i$, which contradicts the irreducible representation of $z = x_1\cdots x_{n+1}$. Hence all b_i's are distinct. Now note $b_1b_2\cdots b_{n+1} = a(x_1\cdots x_{n+1}) = az = z$. The chainwise connected semilattice aS has identity a and is of codimension less than or equal to n; as mentioned before, $\mathrm{Br}(aS) = n$, so that z may be written as the product of a proper subset of $\{b_1,\ldots,b_{n+1}\}$. Without loss of generality, assume $z = b_1\cdots b_n$. Then $z = a(x_1\cdots x_n) = ay_{n+1} = y_{n+1}$. This again contradicts the irreducible representation of z as the product $x_1\cdots x_{n+1}$.

The next lemma is vintage algebraic topology, in the spirit of finite simplicial complexes, and a proof can be constructed from the definition of cohomology groups. For the sake of brevity, we give a proof using Lemma 3.

LEMMA 5. *Let $\{A_1,\ldots,A_{n+1}\}$ be a collection of distinct, compact, acyclic sets such that $\bigcap\{A_i: i \in I\}$ is acyclic and nonempty for each I properly contained in $\{1,\ldots,n+1\}$, but $\bigcap\{A_i: i = 1,2,\ldots,n+1\} = \square$. Then*

$$H^{n-1}(A_1 \cup \cdots \cup A_{n+1}) \neq 0.$$

PROOF. Note that $I \neq J \subseteq \{1,\ldots,n+1\}$ implies

$$\bigcap\{A_i: i \in I\} \neq \bigcap\{A_j: j \in J\}.$$

If $n = 1$, then $\tilde{H}^0(A_1 \cup A_2) \neq 0$, since $A_1 \cap A_2 = \square$. Suppose the conclusion is valid for all families of k sets satisfying the hypotheses, and let $\{A_i\}_{i=1}^{k+1}$ be a family of $k+1$ sets which satisfy the hypotheses. Consider the Mayer-Vietoris sequence:

$$H^{k-2}\left(\bigcup_{i=1}^{k} A_i\right) \times H^{k-2}(A_{k+1}) \to H^{k-2}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$\Delta \swarrow$$

$$H^{k-1}\left(\bigcup_{i=1}^{k+1} A_i\right) \to H^{k-1}\left(\bigcup_{i=1}^{k} A_i\right) \times H^{k-1}(A_{k+1}).$$

By Lemma 3, $\bigcup\{A_i: i = 1,\ldots,k\}$ is acyclic, so that Δ is an isomorphism. For $i = 1,\ldots,k$, set $B_i = A_i \cap A_{k+1}$ and note that if $i \neq j$, then $B_i \neq B_j$. Let $I \subseteq \{1,\ldots,k\}$; then $\bigcap\{B_i: i \in I\} = \bigcap\{A_j: j \in I \cup \{k+1\}\}$. Hence $\bigcap\{B_i: i = 1,\ldots,k\} = \square$ and $\bigcap\{B_i: i \in I\}$ is acyclic (and nonempty) for every I properly contained in $\{1,\ldots,k\}$. By the inductive hypothesis, $H^{k-2}(\bigcup_{i=1}^{k} B_i) \neq 0$, so that $H^{k-1}(\bigcup_{i=1}^{k+1} A_i) \neq 0$.

3. Proof of the theorem. Let X be a compact, chainwise connected topological semilattice, $\operatorname{cd}(X) = n$. Let z be an element of breadth $n+1$ with irreducible representation $z = x_1 x_2 \cdots x_{n+1}$. We first reduce X to more manageable size. Fix $i \in \{1,\ldots,n+1\}$; for $j \neq i$, let C_{ij} be an arc-chain with endpoints x_i and $x_i x_j$. Let $A_i = \Pi\{C_{ij} \mid j \neq i\}$. Note that each A_i contains all finite products of the x_j's which contain x_i as a factor. Finally, let S be the semilattice generated by $\bigcup\{A_i: i = 1,\ldots,n+1\}$. It is easily verified that S is compact, chainwise connected, with maximal elements $x_1,\ldots,x_{n+1}$ and minimal element z. Since $\operatorname{Br}(S) = n+1$, $\operatorname{cd}(S) = n$. We will show that z is an inner point relative to S, and appeal to Lemma 2 to complete the argument. To this end, all further calculations are performed within S.

As in the proof of Lemma 4, let $y_i = x_1 x_2 \cdots \hat{x}_i \cdots x_{n+1}$. Let

$$B = \bigcup \{M(y_i): i = 1,\ldots,n+1\}.$$

By Lemmas 4 and 5, $H^{n-1}(B) \neq 0$. Let $U = S \setminus B$; we show this set satisfies the hypothesis of Lemma 1. For any open set W containing z, $W \subseteq U$, there exists an open set V containing z, $V \subseteq W$, such that $S \setminus V = \bigcup\{M(q_j): j = 1,\ldots,h\}$ **[14]**. As $S \setminus U = B \subseteq S \setminus W \subseteq S \setminus V$, it clearly suffices to show j^*: $H^{n-1}(S \setminus V) \to H^{n-1}(S \setminus U)$ is not trivial.

For ease of notation, let $a = x_1$ henceforth. Let λ_a: $S \to aS$ by $\lambda_a(t) = at$, let $\lambda_1 = \lambda_a \mid S \setminus V$ and $\lambda = \lambda_a \mid B$. The following diagram is independent of path:

$$\begin{array}{ccc} H^{n-1}(a(S\setminus V)) & \xrightarrow{\lambda_1^*} & H^{n-1}(S\setminus V) \\ j_1^* \downarrow & & j^* \downarrow \\ H^{n-1}(aB) & \xrightarrow{\lambda^*} & H^{n-1}(B) \end{array}$$

In order to have $j^* \neq 0$, it suffices to show that λ^* is an isomorphism and $j_1^* \neq 0$.

The former result is accomplished via the Vietoris Mapping Theorem. Fix $x \in aB$; we show $\lambda^{-1}(x)$ is an acyclic subset of B. Clearly $\lambda^{-1}(x) \cap M(y_i) = \lambda_a^{-1}(x) \cap M(y_i)$. Let $\Lambda \subseteq \{2,\ldots,n+1\}$ be defined by $i \in \Lambda$ if and only if $\lambda^{-1}(x) \cap M(y_i) \neq \square$. Thus $\lambda^{-1}(x) = [\lambda^{-1}(x) \cap M(y_1)] \cup [\bigcup_{i\in\Lambda}(\lambda^{-1}(x) \cap M(y_i))]$. Note that the second of the two bracketed sets is, in fact, $\lambda^{-1}(x) \cap (\bigcap_{i\in\Lambda} M(y_i))$. To see this, fix $t \in \bigcup_{i\in\Lambda}(\lambda^{-1}(x) \cap M(y_i))$; it suffices to show $t \geqslant y_i$ for each $i \in \Lambda$. Fix $i \in \Lambda$, let $s \in \lambda^{-1}(x) \cap M(y_i)$. Then, since $i \neq 1$, $a \in M(y_i)$ also, whence $x = as \in M(y_i)$. Thus $y_i \leqslant x = at \leqslant t$. Since $\lambda^{-1}(x) \cap M(y_i) = \lambda_a^{-1}(x) \cap M(y_i)$ and $\lambda_a^{-1}(x)$ is a compact, chainwise connected semilattice, $\lambda^{-1}(x) \cap M(y_i)$ is a principal filter in $\lambda_a^{-1}(x)$ and is therefore acyclic. In a similar manner, $\lambda^{-1}(x) \cap (\bigcap_{i\in\Lambda} M(y_i))$, iff nonempty, is acyclic.

Finally, if both $\lambda^{-1}(x) \cap M(y_1)$ and $\lambda^{-1}(x) \cap (\bigcap_{i\in\Lambda} M(y_i))$ are nonempty, then choose t in the first set, s in the second. Since $x = at = as$, and $a \in \bigcap_{i\in\Lambda} M(y_i)$, we have $x \in \bigcap_{i\in\Lambda} M(y_i)$, from which it follows that $t \in \bigcap_{i\in\Lambda} M(y_i) \cap \lambda^{-1}(x)$. Hence, in this case the first set is a subset of the second, so that the intersection is again acyclic. By Lemma 3, $\lambda^{-1}(x)$ is acyclic; by the Vietoris Mapping Theorem, λ^* is therefore an isomorphism in all dimensions.

We will invoke a similar technique to establish that $j_1^* \neq 0$; however, the use of Lemma 3 is somewhat more delicate than above. Since λ^* is an isomorphism and $H^{n-1}(B) \neq 0$, it suffices to show that j_1^* is surjective. By exactness of the pair sequence $(a(S\setminus V), aB)$, this can be accomplished by showing $H^n(a(S\setminus V), aB) = 0$. From the Strong Excision Theorem **[13]**, decomposing $a(S\setminus V)$ as $[(S\setminus V) \cap L(a)] \cup aB$, $H^n(a(S\setminus V), aB) \cong H^n((S\setminus V) \cap L(a), (S\setminus V) \cap aB)$. We show the latter group to be trivial by showing that each factor is acyclic and again appealing to the exactness of pair sequences.

(i) $a(S\setminus V) = ((S\setminus V) \cap L(a)) \cup aB$. This is a consequence of the construction of S. Let $x = at$, with $t \in S\setminus V$. If $t \in L(a)$, then $x = at = t$; hence it must be shown that, if $t \notin L(a)$, then $at \in aB$. Let $t = t_1 \cdots t_m$, where each t_k belongs to some C_{ij}, an arc-chain with endpoints x_i and x_ix_j. Since $t \notin L(a)$, $i \neq 1$ for each upper endpoint x_i. If, for some j, $x_j = 1$, then $a(x_ix_j) \leqslant at_k \leqslant ax_i$, whence $at_k = ax_i \in aM(y_1)$. If $x_j \neq a$, then $y_1 \leqslant x_ix_j$, so that again $at_k \in aM(y_1)$. Thus $at = at_1 \cdots t_m = (at_1) \cdots (at_m) \in aM(y_1) \subseteq aB$. Right to left inclusion is immediate.

(ii) $(S\setminus V) \cap L(a)$ *is acyclic*. By construction, $S\setminus V$ is the finite union of principal filters; cut down to $L(a)$, a chainwise connected semilattice, they retain this property. The intersection of all of these filters (which meet $L(a)$ at all) must contain a and is therefore itself a principal filter in $L(a)$. By Lemma 3, the set $(S\setminus V) \cap L(a)$ is acyclic.

(iii) $(S\setminus V) \cap aB$ *is acyclic*. If $n = 1$, then $B = \{x_1 = a, x_2\}$, so that $(S\setminus V) \cap aB = \{a\}$ in this case. We assume henceforth that $n \geqslant 2$. For $2 \leqslant j \leqslant n+1$, define

$$T_j = (S\setminus V) \cap a[M(y_1) \cap L(x_j)],$$
$$S_j = \bigcup_{k\neq 1, j} [M(y_k) \cap L(a)], \quad J_j = S_j \cup T_j.$$

A direct argument establishes $(S\setminus V) \cap aB = \bigcup\{J_j\colon j \geqslant 2\}$. We show that the family of J_j's satisfies the hypotheses of Lemma 3. Each S_j is the union of principal filters in $L(a)$, hence is acyclic by Lemma 3. In a similar manner, each T_j is the union of finitely many principal filters in the chainwise connected subsemilattice $a[M(y_1) \cap L(x_j)]$, even though some of the $M(q_i)$'s which comprise $S\setminus V$ will fail to intersect this set. Thus T_j is also acyclic by Lemma 3. The set $S_j \cap T_j$ is, like T_j, the finite union of principal filters in $a[M(y_1) \cap L(x_j)]$, and hence acyclic. Once more applying Lemma 3, we have J_j acyclic. Next fix Λ, a nonempty subset of $\{2,\ldots,n+1\}$, and let $J_\Lambda = \bigcap\{J_j\colon j \in \Lambda\}$, $T_\Lambda = \bigcap\{T_j\colon j \in \Lambda\}$, and $S_\Lambda = \bigcap\{S_j\colon j \in \Lambda\}$. Finally, for $\Lambda' \subseteq \Lambda$, let $F_{\Lambda'} = S_{\Lambda'} \cap T_{\Lambda\setminus\Lambda'}$. It is a tautology that $J_\Lambda = \bigcup\{F_{\Lambda'}\colon \Lambda' \subseteq \Lambda\}$, and that $F_\Lambda \cap F_\Omega = F_{\Lambda\cup\Omega}$ for any subsets $\Lambda, \Omega \subseteq \{2,\ldots,n+1\}$. To prove J_Λ is acyclic, it suffices to prove each $F_{\Lambda'}$ is acyclic, that F_Λ is nonempty, then invoke Lemma 3 again. Now $S_{\Lambda'}$ remains the finite union of principal filters in $L(a)$, whereas $T_{\Lambda\setminus\Lambda'} = (S\setminus V) \cap a[M(y_1) \cap L(w)]$, where $w = \prod\{x_j\colon j \in \Lambda \setminus \Lambda'\}$. Because $wa \leqslant a$, we may therefore regard $F_{\Lambda'} = S_{\Lambda'} \cap T_{\Lambda\setminus\Lambda'}$ as the union of principal filters in the chainwise connected semilattice $a[M(y_1) \cap L(w)]$, and thus $F_{\Lambda'}$ is acyclic by Lemma 3. Since $a \in S_\Lambda = F_\Lambda$, the set F_Λ is nonempty. This establishes that the family $\{F_{\Lambda'}\colon \Lambda' \subseteq \Lambda\}$ satisfies the hypotheses of Lemma 3, so that J_Λ is itself acyclic. Hence the family $\{J_j\colon j \geqslant 2\}$ also satisfies the conditions of Lemma 3, from which we conclude at last that $(S\setminus V) \cap aB$ is acyclic.

From our earlier remarks, $j_1^* \neq 0$, whence $j^* \neq 0$, and the proof of the theorem is complete.

In closing, it is a pleasure to record our indebtedness to our colleague, Michael Friedberg, for his patient listening and valuable comments.

References

1. L. W. Anderson, *On the breadth and co-dimension of a topological lattice*, Pacific J. Math. **9** (1959), 327–333.
2. K. Baker and A. R. Stralka, *Compact, distributive lattices of finite breadth*, Pacific J. Math. **34** (1970), 311–320.
3. D. R. Brown, *Topological semilattices on the two-cell*, Pacific J. Math. **15** (1965), 36–46.
4. H. Cohen, *A cohomological definition of dimension for locally compact Hausdorff spaces*, Duke Math. J. **21** (1954), 209–224.
5. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. Scott, *A compendium of continuous lattices*, Heidelberg, 1980.
6. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Merrill, Columbus, Ohio, 1966.
7. R. J. Koch, *Arcs in partially ordered spaces*, Pacific J. Math. **9** (1959), 723–728.
8. J. D. Lawson, *The relation of breadth and codimension in topological semilattices*, Duke Math. J. **37** (1970), 207–212.
9. ______, *The relation of breadth and codimension in topological semilattices*. II, Duke Math. J. **38** (1971), 555–559.
10. ______, *A generalized version of the Vietoris-Begle theorem*, Fund. Math. **55** (1969), 65–72.
11. J. D. Lawson and B. Madison, *Peripheral and inner points*, Fund. Math. **69** (1970), 253–266.
12. ______, *Peripherality in semigroups*, Semigroup Forum **1** (1970), 128–142.
13. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
14. J. W. Stepp, *Semilattices which are embeddable in a product of min intervals*, Proc. Amer. Math. Soc. **28** (1971), 81–86.
15. A. D. Wallace, *Acyclicity of compact connected semigroups*, Fund. Math. **50** (1961), 99–105.

Department of Mathematics, University of Houston, Houston, Texas 77004