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ON THE COVERING DIMENSION OF INVERSE LIMITS[1]

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ABSTRACT. We give sufficient conditions for an inverse system of spaces of covering dimension $\leqslant n$ to have inverse limit of covering dimension $\leqslant n$.

1. Introduction. Throughout this paper all spaces are assumed to be completely regular T_2 and all maps to be continuous. For a set A, $|A|$ denotes the cardinality of A.

Let $\{X_\alpha, \pi_{\alpha\beta}\}$ (more precisely, $\{X_\alpha, \pi_{\alpha\beta}\}_{\alpha,\beta\in\Omega}$) be an inverse system of spaces over a directed set Ω, and let X be its inverse limit with projections π_α: $X \to X_\alpha$ for α in Ω. Concerning the covering dimension of the inverse limit X, the most natural problem is the following: Under what conditions is it true that

$$\dim X \leqslant \sup_\alpha \dim X_\alpha?$$

The main purpose of this paper is to give solutions of this problem.

THEOREM 1.1. *Let X be the inverse limit of an inverse system $\{X_\alpha, \pi_{\alpha\beta}\}$ over a directed set Ω. If each X_α is an $|\Omega|$-paracompact normal space with* $\dim X_\alpha \leqslant n$ *and each $\pi_{\alpha\beta}$ is a perfect map, then we have* $\dim X \leqslant n$.

COROLLARY 1.2. *If X is the inverse limit of an inverse system $\{X_\alpha, \pi_{\alpha\beta}\}$ consisting of paracompact spaces and perfect maps, then we have*

$$\dim X \leqslant \sup_\alpha \dim X_\alpha.$$

In case Ω is well ordered, we obtain a better one.

THEOREM 1.3. *Let $\{X_\alpha, \pi_{\alpha\beta}\}$ be an inverse system over a well-ordered set Ω. If each X_α is a normal space with* $\dim X_\alpha \leqslant n$, *the inverse limit X is $|\Omega|$-paracompact and each projection π_α is a closed map, then we have* $\dim X \leqslant n$.

Let N be the set of natural numbers directed by the usual order relation; N is a well-ordered set. If Ω is a countable directed set, then we have an order-preserving function f: $N \to \Omega$ such that the image $f(N)$ of f is cofinal in Ω. Therefore the next result follows from Theorem 1.3.

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COROLLARY 1.4. *Let* $\{X_\alpha, \pi_{\alpha\beta}\}$ *be an inverse system over a countable directed set* Ω. *If each* X_α *is a normal space with* $\dim X_\alpha \leqslant n$, *the inverse limit* X *is countably paracompact and each projection* π_α *is a closed map, then we have* $\dim X \leqslant n$.

REMARK. The inverse limit X is $|\Omega|$-paracompact normal in Theorem 1.1, paracompact in Corollary 1.2 and normal in Theorem 1.3 and in Corollary 1.4. (See the proofs of Theorems 1.1 and 1.3.)

Theorems 1.3 and 1.1 will be respectively proved in §§2 and 3. Finally, in §4, we shall give a few applications.

2. Proof of Theorem 1.3. It has been proved by the author in [**4**] that the inverse limit X is normal. To prove that $\dim X \leqslant n$, it is sufficient to prove the following: If $\{U_1, \dots, U_{n+2}\}$ is an open cover of X, then there exists a closed cover $\{F_1, \dots, F_{n+2}\}$ of X such that each $F_i \subset U_i$ and $\bigcap_{i=1}^{n+2} F_i = \varnothing$. (See, e.g., [**8**, Proposition 3.1.6].)

Let $\mathcal{U} = \{U_1, \dots, U_{n+2}\}$ be an open cover of X, and let $U_{i,\alpha}$ be the largest open set of X_α with $\pi_\alpha^{-1}(U_{i,\alpha}) \subset U_i$ for each i and each α. Then $U_i = \bigcup_{\alpha\in\Omega} \pi_\alpha^{-1}(U_{i,\alpha})$ and $\pi_{\alpha\beta}^{-1}(U_{i,\alpha}) \subset U_{i,\beta}$ for each i and for each pair α, β in Ω with $\alpha \leqslant \beta$. Hence the family $\{V_\alpha\}_{\alpha\in\Omega}$, defined by $V_\alpha = \pi_\alpha^{-1}(\bigcup_{i=1}^{n+2} U_{i,\alpha})$ for $\alpha \in \Omega$, is an open cover of X such that $V_\alpha \subset V_\beta$ for each pair $\alpha, \beta \in \Omega$ with $\alpha \leqslant \beta$. Since X is $|\Omega|$-paracompact, there exists a locally finite closed cover $\{H_\alpha\}_{\alpha\in\Omega}$ of X such that $H_\alpha \subset V_\alpha$ for each α (see [**6**, Theorem 5] or [**3**, Lemma 1.1]). Now we shall inductively construct a family $\mathcal{K}_\alpha = \{K_{\alpha,1}, \dots, K_{\alpha,n+2}\}$ of subsets of X_α for each $\alpha \in \Omega$ such that

(1_α) each $K_{\alpha,i}$ is a closed set of X_α with $K_{\alpha,i} \subset U_{i,\alpha}$,

(2_α) $\bigcup_{i=1}^{n+2} K_{\alpha,i} = \pi_\alpha(\bigcup_{\beta<\alpha} H_\beta)$,

(3_α) $\pi_{\beta\alpha}^{-1}(K_{\beta,i}) \cap \pi_\alpha(H_\beta) \subset K_{\alpha,i}$ whenever $\beta \leqslant \alpha$, and

(4_α) $\bigcap_{i=1}^{n+2} K_{\alpha,i} = \varnothing$.

Suppose that $\beta \in \Omega$ and that $\mathcal{K}_\alpha$ has been defined for each $\alpha < \beta$ so that (1_α)–(4_α) are satisfied. Let $\mathcal{K}' = \{K_1', \dots, K_{n+2}'\}$ be the family given by

$$K_i' = \bigcup_{\alpha<\beta} \left(\pi_{\alpha\beta}^{-1}(K_{\alpha,i}) \cap \pi_\beta(H_\alpha)\right) = \pi_\beta\left(\bigcup_{\alpha<\beta} \left(\pi_\alpha^{-1}(K_{\alpha,i}) \cap H_\alpha\right)\right).$$

Since $\{H_\alpha\}_{\alpha\in\Omega}$ is a locally finite family of closed sets of X and π_β is a closed map, each K_i' is a closed set of X_β. It is obvious that each $K_i' \subset U_{i,\beta}$ and $\bigcup_{i=1}^{n+2} K_i' = \pi_\beta(\bigcup_{\alpha<\beta} H_\alpha)$. Suppose that $x \in \bigcap_{i=1}^{n+2} K_i'$. Then for each i there exists $\alpha(i) < \beta$ such that $x \in \pi_{\alpha(i)\beta}^{-1}(K_{\alpha(i),i}) \cap \pi_\beta(H_{\alpha(i)})$. Let $\alpha_0 = \max\{\alpha(1), \dots, \alpha(n+2)\}$, then $\alpha_0 < \beta$ and $\pi_{\alpha_0\beta}(x) \in \pi_{\alpha(i)\alpha_0}^{-1}(K_{\alpha(i),i}) \cap \pi_{\alpha_0}(H_{\alpha(i)})$ for $i = 1, \dots, n+2$. By (3_{α_0}), $\pi_{\alpha_0\beta}(x) \in K_{\alpha_0,i}$ for $i = 1, \dots, n+2$ and hence $\bigcap_{i=1}^{n+2} K_{\alpha_0,i} \neq \varnothing$. This contradicts ($4_{\alpha_0}$). Hence $\bigcap_{i=1}^{n+2} K_i' = \varnothing$. Since X_β is normal, we have a family $\mathcal{G} = \{G_1, \dots, G_{n+2}\}$ of open sets of X_β such that $K_i' \subset G_i \subset U_{i,\beta}$ for each i and $\bigcap_{i=1}^{n+2} G_i = \varnothing$. If we define $\mathcal{W} = \{W_1, \dots, W_{n+2}\}$ by

$$W_i = \left(G_i \cup \left(U_{i,\beta} - \pi_\beta\left(\bigcup_{\alpha<\beta} H_\alpha\right)\right)\right) \cap \pi_\beta(H_\beta),$$

then the family $\mathcal{W}$ is an open cover of $\pi_\beta(H_\beta)$. Since $\pi_\beta(H_\beta)$ is a closed set of X_β, $\pi_\beta(H_\beta)$ is a normal space with $\dim \pi_\beta(H_\beta) \leqslant \dim X_\beta \leqslant n$. Hence, by [**8**, Proposition

3.1.6], we have a closed cover $\{K_1'', \ldots, K_{n+2}''\}$ of $\pi_\beta(H_\beta)$ such that each $K_i'' \subset W_i$ and $\bigcap_{i=1}^{n+2} K_i'' = \varnothing$. Now, let us define $\mathcal{K}_\beta = \{K_{\beta,1}, \ldots, K_{\beta,n+2}\}$ by $K_{\beta,i} = K_i' \cup K_i''$. Then the family $\mathcal{K}_\beta$ satisfies the required conditions. Surely, it is almost obvious that $\mathcal{K}_\beta$ satisfies (1_β)–(3_β). Suppose that $x \in \bigcap_{i=1}^{n+2} K_{\beta,i}$. If $x \in \pi_\beta(\bigcup_{\alpha<\beta} H_\alpha)$ then $x \in \bigcap_{i=1}^{n+2} G_i$, and if $x \notin \pi_\beta(\bigcup_{\alpha<\beta} H_\alpha)$ then $x \in \bigcap_{i=1}^{n+2} K_i''$. In either case, we obtain a contradiction. Hence $\mathcal{K}_\beta$ satisfies condition (4_β). Thus, for each $\alpha \in \Omega$, a family $\mathcal{K}_\alpha$ satisfying (1_α)–(4_α) has been constructed.

Finally, we define $\mathcal{F} = \{F_1, \ldots, F_{n+2}\}$ by

$$F_i = \bigcup_{\alpha \in \Omega} \left(\pi_\alpha^{-1}(K_{\alpha,i}) \cap H_\alpha\right).$$

Then we can prove that the family $\mathcal{F}$ is a closed cover of X such that each $F_i \subset U_i$ and $\bigcap_{i=1}^{n+2} F_i = \varnothing$, by the same argument as previously stated for $\mathcal{K}'$. This completes the proof of Theorem 1.3.

3. Proof of Theorem 1.1. We may suppose that Ω is infinite; if Ω is finite then we have nothing to prove. Let Ξ be the set of nonempty finite subsets of the directed set Ω, directed by set inclusion $\subset$. Then $|\Xi| = |\Omega|$. It is easy to construct an order-preserving function $f\colon \Xi \to \Omega$ such that $f(\{\alpha\}) = \alpha$ for each α in Ω. Let us define an inverse system $\{Y_\xi, \rho_{\xi\zeta}\}$ over Ξ as follows: $Y_\xi = X_{f(\xi)}$ if $\xi \in \Xi$, and $\rho_{\xi\zeta} = \pi_{f(\xi)f(\zeta)}$ if $\xi, \zeta \in \Xi$ and $\xi \subset \zeta$. Then each Y_ξ is a $|\Xi|$-paracompact normal space with $\dim Y_\xi \leqslant n$ and each $\rho_{\xi\zeta}$ is a perfect map. Moreover the inverse limit of the inverse system $\{Y_\xi, \rho_{\xi\zeta}\}$ is identical with that of the given inverse system $\{X_\alpha, \pi_{\alpha\beta}\}$. Therefore it is sufficient to prove the theorem for the case when Ω is the set of nonempty finite subsets of an infinite set.

Now let Λ be an infinite set and let Ω be the set of all nonempty finite subsets of Λ directed by set inclusion. Then $|\Lambda| = |\Omega|$. Suppose that $\{X_\alpha, \pi_{\alpha\beta}\}$ is an inverse system over Ω such that each X_α is a $|\Lambda|$-paracompact normal space with $\dim X_\alpha \leqslant n$ and each $\pi_{\alpha\beta}$ is a perfect map. Let X be its inverse limit with projections $\pi_\alpha\colon X \to X_\alpha$ for $\alpha \in \Omega$, then each π_α is a perfect map (see, e.g., [**8**, Proposition 1.7.5]). Hence, by [**6**, Theorem 16], X is $|\Lambda|$-paracompact (if each X_α is paracompact, then X is paracompact). Hence, by [**4**, Proposition 4.1], X is normal. To prove $\dim X \leqslant n$, we shall use induction on $|\Lambda|$. Let κ be the cardinal (= the initial ordinal) of Λ. Then we may suppose that Λ is the set of ordinals less than κ. (Notice that the order relations in Λ and Ω are denoted by $\leqslant$ and $\subset$ respectively.)

For a moment, let us fix an element λ in Λ. Let $\Lambda(\lambda)$ be the set of ordinals less than or equal to λ, and let $\Omega(\lambda)$ be the set of nonempty finite subsets of $\Lambda(\lambda)$. Since $\Lambda(\lambda)$ is a subset of Λ, $\Omega(\lambda)$ is a directed subset of Ω. Hence $\mathcal{S}_\lambda = \{X_\alpha, \pi_{\alpha\beta}\}_{\alpha,\beta \in \Omega(\lambda)}$ is an inverse system over $\Omega(\lambda)$, and its inverse limit is a normal space by the same reason for X. If we denote by Z_λ the inverse limit of $\mathcal{S}_\lambda$ and denote by π_α^λ the projection from Z_λ onto X_α for each $\alpha \in \Omega(\lambda)$, then naturally we have the unique map $\sigma_\lambda\colon X \to Z_\lambda$ such that $\pi_\alpha^\lambda \circ \sigma_\lambda = \pi_\alpha$ for each $\alpha \in \Omega(\lambda)$. Moreover, the map σ_λ is perfect (of course, closed), because each π_α is perfect (see [**8**, Proposition 2.5.16]). Since $|\Lambda(\lambda)| < |\Lambda|$, we have $\dim Z_\lambda \leqslant n$ by the induction hypothesis.

Thus, for each $\lambda \in \Lambda$, the normal space Z_λ with $\dim Z_\lambda \leqslant n$ has been defined. If $\lambda, \mu \in \Lambda$ and $\lambda \leqslant \mu$, similarly we have the unique map $\sigma_{\lambda\mu}\colon Z_\mu \to Z_\lambda$ such that $\pi_\alpha^\lambda \circ \sigma_{\lambda\mu} = \pi_\alpha^\mu$ for each $\alpha \in \Omega(\lambda)$. Obviously, $\{Z_\lambda, \sigma_{\lambda\mu}\}$ is an inverse system over the well-ordered set Λ such that the inverse limit is exactly (homeomorphic to) X and the projection from X onto Z_λ is σ_λ for each $\lambda \in \Lambda$. Therefore, by Theorem 1.3, we have $\dim X \leqslant n$. This completes the proof of Theorem 1.1.

4. Applications. The following propositions are well known.

PROPOSITION 4.1 (V. L. KLJUŠIN [5]). *If X is a paracompact M-space (i.e., a space which admits a perfect map onto a metrizable space) with* $\dim X \leqslant n$, *then X is (homeomorphic with) the inverse limit of an inverse system $\{X_\alpha, \pi_{\alpha\beta}\}$ such that each X_α is a metrizable space with* $\dim X_\alpha \leqslant n$ *and each $\pi_{\alpha\beta}$ is a perfect map.*

PROPOSITION 4.2 (M. KATĚTOV [2] OR K. MORITA [7]). (a) *If X is the product of a finite family $\{X_1, \ldots, X_m\}$ of nonempty metrizable spaces, then we have*

$$\dim X \leqslant \dim X_1 + \cdots + \dim X_m.$$

(b) *The product of a countable family of strongly zero-dimensional*[2] *metrizable spaces is strongly zero-dimensional.*

The inverse limit X of an inverse system $\{X_\alpha, \pi_{\alpha\beta}\}$ consisting of metrizable spaces and perfect maps is a paracompact M-space, because each projection $\pi_\alpha\colon X \to X_\alpha$ is a perfect map. Therefore the following theorem is a direct consequence of Corollary 1.2 and Proposition 4.1.

THEOREM 4.3. *A space X is a paracompact M-space with* $\dim X \leqslant n$ *if and only if X is (homeomorphic with) the inverse limit of an inverse system $\{X_\alpha, \pi_{\alpha\beta}\}$ such that each X_α is a metrizable space with* $\dim X_\alpha \leqslant n$ *and each $\pi_{\alpha\beta}$ is a perfect map.*

The following theorem is a generalization of Proposition 4.2.

THEOREM 4.4. (a) *If X is the product of a finite family $\{X_1, \ldots, X_m\}$ of nonempty paracompact M-spaces, then we have*

$$\dim X \leqslant \dim X_1 + \cdots + \dim X_m.$$

(b) *The product of a countable family of strongly zero-dimensional paracompact M-spaces is strongly zero-dimensional.*

Theorem 4.4 follows from Propositions 4.1 and 4.2 and Corollary 1.2, together with Lemma 4.5 below.

Let $\{\Omega_\lambda\}_{\lambda\in\Lambda}$ be a family of directed sets. Suppose that for each $\lambda \in \Lambda$ an inverse system $\mathcal{S}_\lambda = \{X_{\alpha_\lambda}, \pi_{\alpha_\lambda\beta_\lambda}\}$ over Ω_λ is given. If Ω is the product of $\{\Omega_\lambda\}_{\lambda\in\Lambda}$, then Ω is directed by the natural order relation $\leqslant$ defined as follows: For $\alpha, \beta \in \Omega$, $\alpha \leqslant \beta$ if and only if $p_\lambda(\alpha) \leqslant p_\lambda(\beta)$ in Ω_λ for each $\lambda \in \Lambda$, where p_λ is the projection from Ω onto Ω_λ for each $\lambda \in \Lambda$. For each $\alpha \in \Omega$ let X_α be the product of the family $\{X_{p_\lambda(\alpha)}\}_{\lambda\in\Lambda}$ of spaces, and for each pair $\alpha, \beta \in \Omega$ with $\alpha \leqslant \beta$ let $\pi_{\alpha\beta}$ be the product

[2] A space X is called *strongly zero-dimensional*, if $\dim X = 0$.

of the family $\{\pi_{p_\lambda(\alpha)p_\lambda(\beta)}\}_{\lambda\in\Lambda}$ of maps. Obviously, $\mathfrak{S} = \{X_\alpha, \pi_{\alpha\beta}\}$ is an inverse system over Ω; we call this inverse system $\mathfrak{S}$ the *product* of the family $\{\mathfrak{S}_\lambda\}_{\lambda\in\Lambda}$ of inverse systems.

LEMMA 4.5. *Let $\mathfrak{S}$ be the product of a family $\{\mathfrak{S}_\lambda\}_{\lambda\in\Lambda}$ of inverse systems.*

(a) *If X_λ is the inverse limit of $\mathfrak{S}_\lambda$ for each $\lambda \in \Lambda$, then the inverse limit X of $\mathfrak{S}$ is homeomorphic with the product of the family $\{X_\lambda\}_{\lambda\in\Lambda}$.*

(b) *If each bonding map of $\mathfrak{S}_\lambda$ is perfect for each $\lambda \in \Lambda$, then each bonding map of $\mathfrak{S}$ is perfect.*

PROOF. (a) Almost obvious.

(b) As is well known, the product of perfect maps is perfect.

REMARK. (a) of Theorem 4.4 is also a direct consequence of the following product theorem due to V. V. Filippov [**1**]:

If the product $X \times Y$ of a paracompact M-space X and a space Y is countably paracompact normal, then we have

$$\dim X \times Y \leqslant \dim X + \dim Y.$$

REFERENCES

1. V. V. Filippov, *On the dimension of product of topological spaces*, Fund. Math. **106** (1980), 181–212.

2. M. Katětov, *On the dimension of non-separable spaces*. I, Czechoslovak Math. J. **2** (1952), 333–368.

3. Y. Katuta, *On the normality of the product of a normal space with a paracompact space*, General Topology Appl. **1** (1971), 295–319.

4. _____, *Paracompactness, collectionwise normality and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku **13** (1977), 165–172.

5. V. L. Kljušin, *Perfect mappings of paracompact spaces*, Dokl. Akad. Nauk SSSR **159** (1964), 734–737 = Soviet Math. Dokl. **5** (1964), 1583–1586.

6. J. Mack, *Directed covers and paracompact spaces*, Canad. J. Math. **19** (1967), 649–654.

7. K. Morita, *Normal families and dimension theory for metric spaces*, Math. Ann. **128** (1954), 350–362.

8. A. R. Pears, *Dimension theory of general spaces*, Cambridge Univ. Press, New York, 1975.

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