A SYMPLECTIC FIXED POINT THEOREM ON OPEN MANIFOLDS

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Abstract. In 1968 Bourgin proved that every measure-preserving, orientation-preserving homeomorphism of the open disk has a fixed point, and he asked whether such a result held in higher dimensions. Asimov, in 1976, constructed counterexamples in all higher dimensions. In this paper we answer a weakened form of Bourgin's question dealing with symplectic diffeomorphisms: every symplectic diffeomorphism of an even-dimensional cell sufficiently close to the identity in the \( C^1 \)-fine topology has a fixed point. This result follows from a more general result on open manifolds and symplectic diffeomorphisms.

Introduction. Fixed point theorems for area-preserving mappings have a history which dates back to Poincaré's "last geometric theorem", i.e., any area-preserving mapping of an annulus which twists the boundary curves in opposite directions has at least two fixed points. More recently it has been proved that any area-preserving, orientation-preserving mapping of the two-dimensional sphere into itself possesses at least two distinct fixed points (see [N, Si]). In the setting of noncompact manifolds, Bourgin [B] showed that any measure-preserving, orientation-preserving homeomorphism of the open two-cell \( B^2 \) has a fixed point. For Bourgin's theorem one assumes that the measure is finite on \( B^2 \) and that the measure of a nonempty open set is positive. Bourgin also gave a counterexample to the generalization of the theorem for the open ball in \( \mathbb{R}^3 \) and asked the question whether his theorem remains valid for the open balls in low dimensions. In [As] Asimov constructed counterexamples for all dimensions greater than two and actually got a flow of measure-preserving, orientation-preserving diffeomorphisms with no periodic points.

To formulate our results and place the comments above into our framework, we need some concepts from symplectic geometry. A smooth manifold is called symplectic if there exists a nondegenerate, closed, differentiable 2-form \( \omega \) defined on \( M \). A differentiable mapping \( f \) of \( M \) into itself is called symplectic if \( f \) preserves the form \( \omega \). We refer to the texts by Abraham and Marsden [A & M] and Arnold [A] for the general background in symplectic geometry.

We reformulate Bourgin's question to ask: does every symplectic mapping of a \( 2n \)-dimensional cell, equipped with a symplectic structure, have a fixed point? Using a generalization of a theorem of Weinstein [W2], we answer this question affirmatively for mappings sufficiently close to the identity.
1. Preliminaries. All manifolds are assumed to be finite-dimensional, $C^\infty$-smooth, and without boundary. A manifold $M$ is open if $M$ has no compact components. Let $\epsilon(M)$ denote the ends of $M$, and let $\tilde{M} = M \cup \epsilon(M)$ be the completion of $M$. We consider manifolds $M$ where the number of ends, denoted by $\epsilon(M)$, is finite and where $\tilde{M}$ has a smooth manifold structure without boundary. For the general problem of completing an open manifold with finitely many ends see Siebenmann’s thesis [S].

If $M$ is a manifold with symplectic form $\omega$, then $\text{Diff}(M, \omega)$ denotes the group of symplectic diffeomorphisms of $M$. The closed one-forms on $M$ are denoted by $Z^1(M)$. Both of these function spaces are topologized with the $C^1$-fine topology. See [H, p. 35] for a good account of the $C^1$-fine topology.

We require the basic formalism of “cotangent co-ordinates” contained in the following theorem of Weinstein.

**Theorem 1.1** [W1, Proposition (2.7.4) or W2, Theorem 7.2]. If $(M, \omega)$ is a symplectic manifold, then there is a $C^1$-fine neighborhood $A \subset \text{Diff}(M, \omega)$ containing the identity map, a $C^1$-fine neighborhood $B \subset Z^1(M)$ containing the zero form, and a homeomorphism $V: A \to B$. If $f \in A$, then a point $x \in M$ is a fixed point of $f$ if and only if $(V(f))(x) = 0$.

**Proof.** If $f$ is in $\text{Diff}(M, \omega)$, then the graph of $f$ is a Lagrangian submanifold of $M \times M$ with the symplectic structure $\pi_1^*\omega - \pi_2^*\omega$, where $\pi_1$ and $\pi_2$ are the projections. There exists a neighborhood $U$ of the diagonal $\Delta(M) = \{(m, m): m \in M\}$ and a bijection of $U$ onto a neighborhood $W$ of the zero-section in $T^*M$, taking Lagrangian submanifolds of $U$ onto Lagrangian submanifolds lying in $W$. If $f$ is close enough to the identity, in the sense that the graph of $f$ is contained in $U$, then there is a one-form $V(f) \in Z^1(M)$ whose image is contained in $W$. Clearly, $f(x) = x$ if and only if $(V(f))(x) = 0$. □

Various fixed point theorems in symplectic geometry result from Theorem 1.1. For examples see [M, N, S, W1, and W2]. Let $M$ be a compact manifold and $\eta$ a closed one-form. Define $c(\eta)$ to be the number of zeros of $\eta$. Define $c(M) = \text{glb} \{c(\eta): \eta \in Z^1(M)\}$. If $M$ is a symplectic manifold with symplectic form $\omega$, then there is a $C^1$-neighborhood of $\text{id}_M$ in $\text{Diff}(M, \omega)$, so that if $f$ is in this neighborhood, then $V(f)$ is a closed one-form. Furthermore, the number of fixed points of $f$ is equal to $c(V(f))$. Now assume $M$ is simply connected, so that every closed one-form is exact. Then $c(M) \geq 2$ since every smooth function on a compact manifold has at least two critical points. Therefore, in this $C^1$-neighborhood of $\text{id}_M$ every $f$ has at least two fixed points.

2. The main theorem. When the manifold $M$ is not compact there are functions with no critical points, and hence there are closed one-forms with no zeros. Therefore, $c(M) = 0$. In this section we extend the fixed point theorem of Weinstein to open symplectic manifolds. Note that while $M$ may be a symplectic manifold, its completion $\tilde{M}$ may carry no symplectic structure at all. In particular, for the open
A SYMPLECTIC FIXED POINT THEOREM

2\(n\)-cell \(B^{2n} = \{ x \in \mathbb{R}^{2n}; \| x \| < 1 \}\) the completion is homeomorphic to \(S^{2n}\), which has no symplectic structure for \(n > 1\). The open manifold \(B^{2n}\) has the standard symplectic structure induced from \(\mathbb{R}^{2n}\).

**Theorem 2.1.** If \((M, \omega)\) is a symplectic manifold with \(c(M) < c(\hat{M})\), then there exists a \(C^1\)-fine neighborhood \(A\) of \(\text{id}_M\) in \(\text{Diff}(M, \omega)\) such that every \(f \in A\) has at least \(c(\hat{M}) - e(M)\) fixed points.

**Proof.** Assume \(M\) is embedded in \(\hat{M}\) as an open submanifold. Let \(\phi: \hat{M} \to \mathbb{R}\) be a nonnegative function vanishing only on the ends of \(M\), \(\phi(x) = 0\) if and only if \(x \in \hat{M} - M\). Let \(B \subset Z^1(M)\) be the set of one-forms defined by \(\phi\),

\[ B = \{ \eta \in Z^1(M): \| \eta(x) \| < \phi(x), \| D\eta(x) \| < \phi(x) \} \]

where the norms arise from a riemannian metric on \(\hat{M}\). So \(B\) is an open subset and every \(\eta \in B\) extends to a form \(\tilde{\eta}\) on \(\hat{M}\) such that \(\tilde{\eta}(x) = 0\) for \(x \in \hat{M} - M\). By taking an intersection, if necessary, we may assume that \(B\) satisfies the conclusions of Theorem 1.1. Since \(c(\hat{M}) - e(\hat{M}) > 0\) and \(c(\tilde{\eta}) \geq c(M)\), it follows that \(c(\tilde{\eta}) - e(M) > 0\), so that \(\tilde{\eta}\) has more zeros than there are points in \(\hat{M} - M\). Therefore \(\eta(x) = 0\) for some \(x \in M\). Now we use Theorem 1.1 to get a \(C^1\)-fine neighborhood \(A\) in \(\text{Diff}(M, \omega)\) containing the identity and a homomorphism \(V: A \to B\). For \(f \in A\), the one-form \(V(f)\) is in \(B\) and so \(f\) has a fixed point \(x \in M\). \(\square\)

We now restrict our attention to manifolds \(M\) diffeomorphic to \(\mathbb{R}^{2n}\). Let \(\omega\) be any symplectic structure on \(M\). Clearly, \(e(M) = 1\) and by picking a point \(N \in S^{2n}\), we can embed \(M\) onto \(S^{2n} - \{N\}\), so that \(\hat{M} \approx S^{2n}\). With this construction and the fact that \(c(S^{2n}) = 2\), we have

**Corollary 2.2.** Let \((M, \omega)\) be a symplectic manifold where \(M\) is diffeomorphic to \(\mathbb{R}^{2n}\). Then there is a neighborhood in the \(C^1\)-fine topology of \(\text{Diff}(M, \omega)\) which contains \(\text{id}_M\), such that every mapping in this neighborhood has a fixed point.

One should be aware that there are symplectic diffeomorphisms of \(\mathbb{R}^{2n}\) with symplectic structure \(\sum dx_i \wedge dy_i\) that have no fixed points, in particular the translations, but there are \(C^1\)-fine neighborhoods of the identity containing no translations. Let \(\phi: \mathbb{R}^{2n} \to \mathbb{R}^+\) be a function vanishing at infinity and use \(\phi\) to define an open neighborhood consisting of the diffeomorphisms \(f\) such that \(\| f(x) - x \| < \phi(x)\) and \(\| Df(x) - I \| < \phi(x)\) for all \(x \in \mathbb{R}^{2n}\).

**References**


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