ON THE DISTRIBUTION OF SELF-NUMBERS

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ABSTRACT. Self-numbers are those integers which cannot be expressed as \( a + f(a) \), where \( f(a) \) denotes the sum of the digits of \( a \) in a given scale. Here I prove that the number of self-numbers less than or equal to a large number \( x \) equals \( Lx + O(\log^2 x) \), where \( L \) is a strictly positive constant.

1. Introduction. Many papers have been devoted to "digitaddition sequences", i.e. sequences \( \{a_n\} \) where \( a_{n+1} = a_n + f(a_n) \), and \( f(a) \) is the sum of the digits of \( a \) in a fixed scale (see for example [1, 2, 3]). In particular self-numbers, numbers that cannot be written in the form \( n + f(n) \), have been studied, and it has been proved, for example, that there are infinitely many of them in any scale (see [2]).

However only a few papers deal with asymptotic results; here the following theorem is proved.

**Theorem.** Let \( A(x) \) denote the number of self-numbers (in the scale of 2) less than or equal to \( x \). Then we have the formula

\[
A(x) = Lx + O((\log x)f([x]))
\]

where \( L > 0 \), \( f(a) \) is as above, and the constant implicit in the \( O \) is effectively computable.

An analogous theorem holds in any scale, the proof being a little more complicated but completely similar.

2. Notation. If \( S \) is any finite set we denote the number of its elements by \( |S| \). We denote by \([x]\) the largest integer less than or equal to \( x \).

If \( a = \Sigma_j a_j2^j \) is the expression of \( a \) in the binary scale we put \( f(a) = \Sigma_j a_j \).

Finally we set \( A'_j(x) = \{n \leq x: n = r + f(r) \text{ has exactly } j \text{ solutions}\} \) and \( A_j(x) = |A'_j(x)| \). It is clear that \( A_0(x) \) is the number of self-numbers less than or equal to \( x \).

3. Proof of the theorem. We separate the proof in two parts: in the first one, we give only a lower bound for \( A_0(x) \) of the type \( A_0(x) > Dx \), with \( D > 0 \), and in the second one we prove \( A_0(x) = Lx + O(\log x)f([x])) = Lx + O(\log^2 x) \). By the first part, \( L \) must be \( > 0 \) and so the demonstration will be complete.

Let \( B(n) = \{a: a + f(a) = n\} \). We have \( B(n) \cap B(m) = \emptyset \) when \( n \neq m \), and \( \bigcup_{n \leq x} B(n) \subset [1, x] \) and so \( \Sigma_{n \leq x} |B(n)| \leq [x] \).
But $\sum_{n<x} |B(n)| = A_1(x) + 2A_2(x) + 3A_3(x) + \ldots$, and comparing these results with the obvious equation

$$[x] = A_0(x) + A_1(x) + A_2(x) + \ldots$$

we obtain

(1) $A_0(x) \geq A_2(x) + 2A_3(x) + 3A_4(x) + \ldots$.

We shall now construct many numbers $n$ such that $|B(n)|$ is at least 2, hence, using inequality (1), we will obtain the desired lower bound.

We note that $\bigcup_{j \geq 2} A'_j(x)$ is not empty if $x \geq 5$ (in fact we have for instance $5 = 3 + f(3) = 4 + f(4)$). Pick then $n_0 \in \bigcup_{j \geq 2} A'_j(x)$ (for example, $n_0 = 5$). Then there exist $a, a'$ with $a \neq a'$ and

$$n_0 = a + f(a) = a' + f(a').$$

Let $k$ be so large that $2^k > \max(a, a')$ and set

$$a_n = a + 2^kn \quad a'_n = a' + 2^kn.$$ 

Then it is clear that $f(a_n) = f(a) + f(2^kn)$ and similarly for $a'$.

Let us consider the sets

$$C(y) = \{n_0 + 2^kn + f(2^kn), n \leq y\} = \{m_1, m_2, \ldots, m_c\}$$

with $c = c(y) = |C(y)|$.

Suppose that $m_j = n_0 + 2^kn_{j_0} + f(2^kn_{j_0})$, $v = 1, 2, \ldots, r_j$. We have $[y] = r_1 + r_2 + \cdots + r_c$. We show that $|B(m_j)|$ is at least $2r_j$ (in particular $|B(m_j)| \geq 2$).

In fact $m_j = a_{j_0} + f(a_{j_0}) = a'_{j_0} + f(a'_{j_0})$, $v = 1, 2, \ldots, r$, and the $a_s, a'_s$ are all distinct because $a, a' < 2^k$.

It follows that $|B(m_1)| + |B(m_2)| + \cdots + |B(m_c)| \geq 2[y]$ and setting $H_b = \{|i \leq c: |B(m_i)| = b\} |$, we have $\sum_b bH_b \geq 2[y]$. 

Set $M = \max(m_j)$. Then $m_j \in A'_{\{B(m_j)\}}(M)$ and we obtain

$$\sum_{b \geq 2} (b - 1)A_b(M) \geq \sum_b (b - 1)H_b \geq \frac{1}{2} \left( \sum_b bH_b \right) \geq [y].$$

Now inequality (1) implies

(2) $A_0(M) \geq [y]$.

From the trivial inequality $f(a) \leq \log 2a/\log 2$ we obtain

$$m_j \leq n_0 + 2^ky + \log 2^{k+1}y/\log 2 \leq Ky$$

if $K$ is sufficiently large, so that by (2) we have $A_0(Ky) \geq [y]$ and we conclude that there exists a constant $D > 0$ such that $A_0(y) \geq Dy$. This completes the first part.

We shall now obtain a formula which says that $A_0(x)$ is "almost additive" in a sense that will be clear in a moment.

Fix an integer $k$ and choose $n$ such that

(3) $2^k + k + 2 \leq n < 2^{k+1}$.

We want to prove that $n$ is a self-number if and only if $n' = n - 2^k - 1$ is (cf. [2]).

In fact let $n' = a + f(a)$. Then $a \leq 2^k - 1$ and it follows that $f(a + 2^k) = f(a) + 1$ and $n = n' + 2^k + 1 = a + 2^k + f(a + 2^k)$. 


Conversely suppose that \( n = a + f(a) \). Then
\[
f(a) \leq \log 2a/\log 2 < \log 2^{k+2}/\log 2 = k + 2
\]
whence \( 2^k < a < 2^{k+1} \).

We obtain \( f(a - 2^k) = f(a) - 1 \) and \( n' = a - 2^k + f(a - 2^k) \), whence \( n \) is a self-number if \( n' \) is.

What we have proved means that if \( m \) satisfies (3) then
\[
A_0(m) - A_0(2^k + k + 2) = A_0(m - 2^k - 1) - A_0(k + 1)
\]
and so
\[
(4) \quad A_0(m) = A_0(m - 2^k) + A_0(2^k) + O(k)
\]
and the same estimate obviously holds also when \( 2^k \leq m \leq 2^k + k + 2 \).

From (4) it follows in particular that
\[
(5) \quad A_0(2^{k+1}) = 2A_0(2^k) + O(k)
\]
and iteration of this formula gives
\[
(6) \quad A_0(2^{k+t}) = 2^tA_0(2^k) + O\left(\sum_{r=1}^{s} 2^r(k + s - r)\right).
\]

Now we have
\[
\sum_{r=1}^{s} 2^r k + \sum_{r=1}^{s} 2^r(s - r) = (2^{r+1} - 2)k + \sum_{r=0}^{s-1} 2^{r+1}
= O\left(2^s k + 2^s \sum_{r=0}^{\infty} t 2^{-t}\right) = O(2^s k + 2^s t) = O(2^s k)
\]
whence after division by \( 2^{k+s} \) (6) becomes
\[
(7) \quad A_0(2^{k+s})/2^{k+s} = A_0(2^k)/2^k + O(k/2^k).
\]

This implies in particular that the sequence \( A_0(2^r)/2^r \) is Cauchy, so that \( A_0(2^r)/2^r \to L \). Now, passing to the limit for \( s \to \infty \) in (7) we deduce that \( A_0(2^k) = L2^k + O(k) \).

Now, making use of (4) we may write, when \( m = \sum_{j=0}^{k} a_j 2^j \),
\[
A_0(m) = \sum_{j=0}^{k} A_0(a_j 2^j) + O(kf(m)) = \sum_{j=0}^{k} a_j A_0(2^j) + O((\log m) f(m))
= L\sum_{j=0}^{k} a_j 2^j + O\left(\sum_{j=0}^{k} a_j j\right) + O((\log m) f(m)) = Lm + O((\log m) f(m))
\]
and the theorem is proved.

4. Remarks. When a particular scale has been chosen one can avoid the use of the first part; in fact if \( L = 0 \) we should have \( A_0(m) \leq K(\log m) f(m) < K'\log m \) for some explicit \( K' \). In case this is false, its falsity can be verified with a finite number of operations, so that the result in the first part becomes really essential only if one deals with the general case.
We note also that the constant \( L \) has not been determined, but, using the theorem with an explicit constant in the \( O \), one may obviously calculate it with any degree of accuracy. I will give here some examples.

From the proof of our theorems it follows easily that, when the scale is \( g \), one has the estimate

\[
\left| \frac{A_0(m)}{m} - L \right| \leq 2 \frac{\log mg^3}{\log g} f(m) \frac{1}{m}
\]

where of course \( L \) depends on \( g \).

Here are some results when \( g = 2, 4, 10 \). (I have not made computations for odd \( g \), since in this case it is not difficult to show that self-numbers are precisely the odd numbers. I omit for simplicity the easy proof.)

\[
\begin{array}{ccc}
g = 2 & \\
m & A_0(m) & A_0(m)/m \\
262,144 & 66,237 & 0.252674 \\
524,288 & 132,470 & 0.252666 \\
786,432 & 198,704 & 0.252665 \\
1,048,576 = 2^{20} & 264,938 & 0.252665 \\
\end{array}
\]

\[
\begin{array}{ccc}
g = 4 & \\
m & A_0(m) & A_0(m)/m \\
262,144 & 54,917 & 0.209491 \\
524,288 & 109,827 & 0.209478 \\
786,432 & 164,737 & 0.209473 \\
1,048,576 & 219,647 & 0.209471 \\
\end{array}
\]

\[
\begin{array}{ccc}
g = 10 & \\
m & A_0(m) & A_0(m)/m \\
250,000 & 24,451 & 0.097804 \\
500,000 & 48,896 & 0.097792 \\
750,000 & 73,340 & 0.097787 \\
1,000,000 & 97,785 & 0.097785 \\
\end{array}
\]

Now let us set in formula (8) \( m = 1,048,576 \) when \( g = 2 \) or 4 and \( m = 1,000,000 \) when \( g = 10 \). In these cases \( f(m) = 1 \) and we easily obtain

\[
\left| \frac{A_0(m)}{m} - L \right| < 10^{-4}
\]

whence \( L = 0.252, 0.209, 0.097 \) when \( g = 2, 4, 10 \) respectively, and where the values are correct to three decimals.
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REFERENCES


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