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ON HOLOMORPHIC FUNCTIONS SATISFYING $|f(z)|(1-|z|^2) \leqslant 1$ IN THE UNIT DISC

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Dedicated to Professor Dr. H.-J. Kowalsky on the occasion of his 60th birthday

ABSTRACT. Let f be holomorphic in $D=\{z \mid |z|<1\}$, $|f(z)|(1-|z|^2) \leqslant 1$ in D, $\overline{\lim}_{|z| \to 1} |f(z)|(1-|z|^2) < 1$ and $L(f) := \{z \mid |f(z)|(1-|z|^2)=1\}$. It is shown that the set $L(f)$ consists of one simple closed curve γ and a finite number of points in the bounded component of $\mathbf{C} \setminus \gamma$ if $L(f)$ is an infinite set.

Let

$$D := \{z \mid |z| < 1\}$$

and

$$R := \left\{f \mid f \text{ holomorphic in } D \text{ and } |f(z)|\left(1-|z|^2\right) \leqslant 1 \text{ for any } z \in D\right\}.$$

In the sequel we shall consider the sets

$$L(f) := \left\{z \mid z \in D \text{ and } |f(z)|\left(1-|z|^2\right) = 1\right\}$$

for $f \in R$.

Cima and Wogen proved in [2]:

THEOREM A. *If $f \in R$, $\lim_{|z| \to 1-0} |f(z)|(1-|z|^2) = 0$ and $L(f)$ is an infinite set, then there are simple closed pairwise disjoint analytic curves $\gamma_1, \ldots, \gamma_k$, $k \geqslant 1$, and points $z_1, \ldots, z_j$, $j \geqslant 0$, such that*

$$L(f) = \bigcup_{i=1}^{k} \gamma_i \cup \{z_1, \ldots, z_j\}.$$

The proof of this theorem was based on a real analytic version of the Weierstrass Preparation Theorem.

It is our aim to give an alternate proof of Theorem A and to show that an infinite set $L(f)$ under these circumstances contains only one simple closed curve and possibly a finite number of points.

First we prove a local version of Theorem A.

THEOREM 1. *If $f \in R$ and $L(f)$ has an accumulation point s, $s \in D$, then there exists a neighbourhood U of s such that $U \cap L(f)$ is a simple arc that does not end in s.*

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PROOF. We consider the function

$$F(z, \bar{z}) := f(z)\,\overline{f(z)}\,(1 - z\bar{z})^2.$$

If $z \in L(f)$, then F has a local maximum and $f(z) \neq 0$. Therefore we get

$$(1) \qquad f'(z)(1 - z\bar{z}) - 2\bar{z}f(z) = 0 \quad \text{for any } z \in L(f).$$

Since $zf'(z) + 2f(z) \not\equiv 0$ in D the function

$$(2) \qquad h(z) := \frac{f'(z)}{zf'(z) + 2f(z)}$$

is meromorphic in D and

$$(3) \qquad h(z) = \bar{z} \quad \text{on } L(f).$$

Let $\{z_k \mid k \in \mathbf{N}\} \subset L(f)$ be a sequence of distinct points with $\lim_{k \to \infty} z_k = s$. It follows from (3) that

$$h(s) = \bar{s}, \quad \overline{h\left(\overline{h(z_k)}\right)} = z_k \quad \text{for } k \in \mathbf{N}.$$

The function $\overline{h(\overline{h(z)})}$ is well defined and holomorphic in a suitable neighbourhood V of s. Hence, using the identity principle, we get

$$(4) \qquad \overline{h\left(\overline{h(z)}\right)} = z \quad \text{on } V.$$

Differentiation yields

$$(5) \qquad h'\left(\overline{h(z)}\right)\overline{h'(z)} = 1 \quad \text{on } V.$$

We now consider the sets

$$M := \{z \mid h(z) = \bar{z}\} \cap V$$

and

$$h'^{-1}(\partial D) := \{z \mid |h'(z)| = 1\}.$$

The relation

$$(6) \qquad M \subset h'^{-1}(\partial D)$$

is an immediate consequence of (5).

If $h''(s) \neq 0$ then h' is univalent in a neighbourhood U of s, $U \subset V$, $h'^{-1}(\partial D) \cap U$ is a simple arc not ending in s and we see, using (5) and (6), that this arc is just the set $M \cap U$.

If there is an $m \in \mathbf{N}$, $m \geqslant 2$, such that s is a zero of h'' of order $m - 1$, then there exist a neighbourhood U of s, $U \subset V$, and a neighbourhood W of the point $h'(s)$ for which the following conditions are fulfilled:

(1) h is univalent in U.

(2) For any $w \in W \setminus \{h'(s)\}$ there are distinct points $z_1, \ldots, z_m \in U$ such that $h'(z_i) = w$, $i = 1, \ldots, m$.

(3) $h'^{-1}(\partial D) \cap U$ consists of $2m$ arcs $\gamma_0, \ldots, \gamma_{2m-1}$ that begin in the point s, $\gamma_i \cap \gamma_j = \{s\}$ for $i \neq j$. The indices may be chosen such that the angle between γ_i and γ_{i+1}, $i = 0, \ldots, 2m - 2$, resp., between γ_{2m-1} and γ_0, is π/m.

These conditions together with (5) yield

$$h\left(h'^{-1}(\partial D) \cap U\right) = \left\{z \mid \bar{z} \in h'^{-1}(\partial D) \cap U\right\}. \tag{7}$$

Since s is an accumulation point of $L(f)$, and therefore of M, there is one arc, say γ_0, that contains an infinite number of points of M. It is a consequence of condition (3) and (7) that this implies $\gamma_0 \subset M \cap U$. Using again (7) and the fact that the conformal map $h|_U$ preserves orientation we get $\gamma_0 \cup \gamma_m = M \cap U$.

In both cases we get a simple arc γ such that $\gamma = M \cap U \subset L(f) \cap U$. It remains to prove that for any $z \in \gamma$, $|f(z)|(1 - z\bar{z}) = 1$. To see this we solve (2) as a differential equation for f by integration along γ from s to z:

$$|f(z)| = \left| f(s) \exp\left(\int_s^z \frac{2h(\zeta)}{1 - \zeta h(\zeta)} d\zeta \right) \right|$$

$$= \frac{1}{1 - s\bar{s}} \exp\left(\operatorname{Re} \int_s^z \frac{2\bar{\zeta}}{1 - \zeta\bar{\zeta}} d\zeta \right) = \frac{1}{1 - z\bar{z}}.$$

REMARK. Theorem A immediately follows from Theorem 1 and its proof.

THEOREM 2. *If $f \in R$, $\overline{\lim}_{z \to 1-0} |f(z)|(1 - |z|^2) < 1$ and $L(f)$ is an infinite set, then there is a simple closed curve γ and points $z_1, \ldots, z_j$, $j \geqslant 0$, such that*

$$L(f) = \gamma \cup \{z_1, \ldots, z_j\}.$$

The points $z_1, \ldots, z_j$ lie in the simply-connected bounded domain G defined by $\partial G = \gamma$, $G \subset D$. There are real numbers $\alpha_{\nu\mu}$, ν, $\mu = 0, \ldots, 2r$, such that

$$\sum_{\nu,\mu=0}^{2r} \alpha_{\nu\mu} x^\nu y^\mu = 0 \quad \textit{for any } z = x + iy \in L(f).$$

PROOF. According to Theorem 1 $L(f)$ contains at least one simple closed curve γ which may be oriented such that $n(\gamma, z) = 1$ for $z \in G$. If we define h by (2) we get from (3),

$$0 < |G| = \frac{1}{2i} \int_\gamma \bar{z} dz = \frac{1}{2i} \int_\gamma h(z)\, dz = \pi \sum_{z \in G} \operatorname{Res} h(z). \tag{8}$$

This means that h has r poles in G, where $r \geqslant 1$ and the poles are counted according to their order.

We now consider the Riemann surface $h(G)$.

The boundary points of $h(G)$ are the points of

$$\gamma^* := \{z \mid \bar{z} \in \gamma\} = h(\gamma).$$

Let G^* be defined by $\partial G^* = \gamma^*$, $G^* \subset D$. Then for $a \in G^*$ we get $n(h(\gamma), a) = -1$ and for $b \in \overline{\mathbf{C}} \setminus \overline{G^*}$ $n(h(\gamma), b) = 0$. It is a consequence of the argument principle that h takes the values $a \in G^*$ $r - 1$ times and the values $b \in \overline{\mathbf{C}} \setminus \overline{G^*}$ r times in G. Using continuity arguments it is easily seen that h takes any value $c \in \gamma^*$ $r - 1$ times in G and r times in $\overline{G}$.

On the Riemann surface $h(G)$ the inverse function of h, h^{-1}, is defined. The analytic continuation of h^{-1} across γ^* is delivered by (4) which yields the continuation

$$h^{-1}(z) = \overline{h(\bar{z})} =: g(z) \tag{9}$$

into $\overline{G^*}$. g is meromorphic in $\overline{G^*}$, $g(\overline{G^*}) \cup G$ forms the Riemann surface of h wherein the analytic continuation of h across γ is defined by $h = g^{-1}$. For later use we mention

$$h(D \setminus \overline{G}) \subset G^*. \tag{10}$$

The above considerations together with standard arguments for algebraic functions (compare for instance [**1**, V. 4]) show that h as well as h^{-1} are algebraic functions of order r. Therefore we get an irreducible algebraic equation

$$\sum_{k=0}^{r} \sum_{l=0}^{t} b_{kl} h^k z^l = 0, \qquad b_{kl} \in \mathbf{C}, \tag{11}$$

to determine the analytic continuations of h.

(4) and (11) show that

$$\sum_{k=0}^{r} \sum_{l=0}^{t} b_{kl} \bar{z}^k \bar{h}^l = 0.$$

Because of the irreducibility of (11) this implies the existence of $\sigma \in \mathbf{C} \setminus \{0\}$ such that for any pair (k, l), $0 \leqslant k \leqslant r$, $0 \leqslant l \leqslant t$,

$$b_{kl} = \sigma \overline{b_{lk}}\,. \tag{12}$$

Hence $b_{kl} = \sigma\bar{\sigma} b_{kl}$, $\sigma = e^{2i\tau}$ and $r = t$. With $a_{kl} := e^{-i\tau} b_{kl}$ we get instead of (10) and (12)

$$\sum_{k=0}^{r} \sum_{l=0}^{r} a_{kl} h^k z^l = 0, \qquad a_{kl} = \overline{a_{lk}}\,. \tag{13}$$

According to (3) we have

$$\gamma \subset L(f) \subset \left\{ z \mid \sum_{k,l=0}^{r} a_{kl} \bar{z}^k z^l = 0 \right\}.$$

This proves the last assertion of Theorem 2 if we identify

$$\sum_{k,l=0}^{r} a_{kl} (x - iy)^k (x + iy)^l = \sum_{\nu,\mu=0}^{2r} \alpha_{\nu\mu} x^\nu y^\mu, \qquad \alpha_{\nu\mu} \in \mathbf{R}.$$

The proof of Theorem 1 shows that for the function h defined in D by (2) there are, in a certain neighbourhood of γ, no points z for which $h(z) = \bar{z}$ except the points of γ. Assuming the existence of a second closed simple curve γ', $\gamma' \subset L(f)$, which defines a bounded domain G' by $\partial G' = \gamma'$, $G' \subset D$, we therefore get three possibilities: (1) $G' \subset D \setminus \overline{G}$, (2) $\gamma \subset G'$, (3) $\gamma' \subset G$.

In case (1) $|\,G'\,| > 0$ implies according to (3) and (8) that h has at least one pole in $G' \subset D \setminus \overline{G}$ which contradicts (10).

In case (2) the inequality $|G'| > |G|$ implies the existence of a pole of h in $G' \setminus \overline{G}$ which leads to the same contradiction.

Case (3) may be treated as case (2) interchanging the roles of γ and γ'.

It remains to show that the points $z_1, \ldots, z_j$ lie in G. This is an immediate consequence of (3) and (10) since $G^* = \{z \mid \bar{z} \in G\}$.

REMARK. Theorem 2 answers in part the question of Cima and Wogen in **[2]** whether $L(f)$ is a connected set.

After the completion of the manuscript we found that meromorphic functions which fulfill the condition $h(z) = \bar{z}$ on a curve are known as Schwarz functions (compare P. Davis, *The Schwarz function and its applications*, Carus Math. Monographs, no. 17, Math. Assoc. Amer., Washington, D.C., 1974).

Another method to show some parts of the proof of Theorem 2 is given in the paper of D. Aharonov and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.

In a forthcoming paper St. Ruscheweyh and the present author will give further details on the construction of functions described in Theorem 2.

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