ON HOLOMORPHIC FUNCTIONS SATISFYING
\[ |f(z)| (1 - |z|^2) \leq 1 \] IN THE UNIT DISC

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Dedicated to Professor Dr. Il.-J. Kowalsky
on the occasion of his 60th birthday

Abstract. Let \( f \) be holomorphic in \( D = \{ z | |z| < 1 \} \), \( |f(z)| (1 - |z|^2) \leq 1 \) in \( D \), \( \lim_{|z| \to 1^{-}} |f(z)| (1 - |z|^2) < 1 \) and \( L(f) := \{ z | |f(z)| (1 - |z|^2) = 1 \} \). It is shown that the set \( L(f) \) consists of one simple closed curve \( \gamma \) and a finite number of points in the bounded component of \( \mathbb{C} \setminus \gamma \) if \( L(f) \) is an infinite set.

Let
\[ D := \{ z | |z| < 1 \} \]
and
\[ R := \{ f | f \text{ holomorphic in } D \text{ and } |f(z)| (1 - |z|^2) \leq 1 \text{ for any } z \in D \} \].

In the sequel we shall consider the sets
\[ L(f) := \{ z | z \in D \text{ and } |f(z)| (1 - |z|^2) = 1 \} \]
for \( f \in R \).

Cima and Wogen proved in [2]:

**Theorem A.** If \( f \in R \), \( \lim_{|z| \to 1^{-}} |f(z)| (1 - |z|^2) = 0 \) and \( L(f) \) is an infinite set, then there are simple closed pairwise disjoint analytic curves \( \gamma_1, \ldots, \gamma_k \), \( k \geq 1 \), and points \( z_1, \ldots, z_j, j \geq 0 \), such that
\[ L(f) = \bigcup_{i=1}^{k} \gamma_i \cup \{ z_1, \ldots, z_j \} \).

The proof of this theorem was based on a real analytic version of the Weierstrass Preparation Theorem.

It is our aim to give an alternate proof of Theorem A and to show that an infinite set \( L(f) \) under these circumstances contains only one simple closed curve and possibly a finite number of points.

First we prove a local version of Theorem A.

**Theorem 1.** If \( f \in R \) and \( L(f) \) has an accumulation point \( s \), \( s \in D \), then there exists a neighbourhood \( U \) of \( s \) such that \( U \cap L(f) \) is a simple arc that does not end in \( s \).
Proof. We consider the function
\[ F(z, \bar{z}) := f(z) \overline{f(z)} (1 - z\bar{z})^2. \]
If \( z \in L(f) \), then \( F \) has a local maximum and \( f(z) \neq 0 \). Therefore we get
\[ f'(z)(1 - z\bar{z}) - 2\bar{z}f(z) = 0 \quad \text{for any } z \in L(f). \]
Since \(zf'(z) + 2f(z) \equiv 0 \) in \( D \) the function
\[ h(z) := \frac{f'(z)}{zf'(z) + 2f(z)} \]
is meromorphic in \( D \) and
\[ h(z) = \bar{z} \quad \text{on } L(f). \]
Let \( \{z_k \mid k \in \mathbb{N}\} \subset L(f) \) be a sequence of distinct points with \( \lim_{k \to \infty} z_k = s \). It follows from (3) that
\[ h(s) = \bar{s}, \quad h(h(z_k)) = z_k \quad \text{for } k \in \mathbb{N}. \]
The function \( h(h(z)) \) is well defined and holomorphic in a suitable neighbourhood \( V \) of \( s \). Hence, using the identity principle, we get
\[ h(h(z)) = z \quad \text{on } V. \]
Differentiation yields
\[ h'(h(z))h'(z) = 1 \quad \text{on } V. \]
We now consider the sets
\[ M := \{z \mid h(z) = z\} \cap V \]
and
\[ h'^{-1}(\partial D) := \{z \mid \left|h'(z)\right| = 1\}. \]
The relation
\[ M \subset h'^{-1}(\partial D) \]
is an immediate consequence of (5).
If \( h''(s) \neq 0 \) then \( h' \) is univalent in a neighbourhood \( U \) of \( s \), \( U \subset V \), \( h'^{-1}(\partial D) \cap U \) is a simple arc not ending in \( s \) and we see, using (5) and (6), that this arc is just the set \( M \cap U \).
If there is an \( m \in \mathbb{N}, m \geq 2 \), such that \( s \) is a zero of \( h'' \) of order \( m - 1 \), then there exist a neighbourhood \( U \) of \( s \), \( U \subset V \), and a neighbourhood \( W \) of the point \( h'(s) \) for which the following conditions are fulfilled:
(1) \( h \) is univalent in \( U \).
(2) For any \( w \in W \setminus \{h'(s)\} \) there are distinct points \( z_1, \ldots, z_m \in U \) such that \( h'(z_i) = w, i = 1, \ldots, m \).
(3) \( h'^{-1}(\partial D) \cap U \) consists of \( 2m \) arcs \( \gamma_0, \ldots, \gamma_{2m-1} \) that begin in the point \( s \), \( \gamma_i \cap \gamma_j = \{s\} \) for \( i \neq j \). The indices may be chosen such that the angle between \( \gamma_i \) and \( \gamma_{i+1}, i = 0, \ldots, 2m - 2 \), resp., between \( \gamma_{2m-1} \) and \( \gamma_0 \), is \( \pi/m \).
These conditions together with (5) yield

\[(7) \quad h(\mathcal{D}(\partial D) \cap U) = \{z \mid \bar{z} \in h^{-1}(\partial D) \cap U\}\]

Since \(s\) is an accumulation point of \(L(f)\), and therefore of \(M\), there is one arc, say \(\gamma_0\), that contains an infinite number of points of \(M\). It is a consequence of condition (3) and (7) that this implies \(\gamma_0 \subset M \cap U\). Using again (7) and the fact that the conformal map \(h |_U\) preserves orientation we get \(\gamma_0 \cup \gamma_m = M \cap U\).

In both cases we get a simple arc \(\gamma\) such that \(\gamma = M \cap U \subset L(f) \cap U\). It remains to prove that for any \(z \in \gamma\), \(|f(z)|(1 - \bar{z}z) = 1\). To see this we solve (2) as a differential equation for \(f\) by integration along \(\gamma\) from \(s\) to \(z\):

\[
|f(z)| = \left| f(s) \exp \left( \int_s^z \frac{2h(\xi)}{1 - \bar{\xi}h(\xi)} \, d\xi \right) \right| = \frac{1}{1 - ss} \exp \left( \Re \int_s^z \frac{2\xi}{1 - \bar{\xi}\xi} \, d\xi \right) = \frac{1}{1 - \bar{z}z}.
\]

**Remark.** Theorem A immediately follows from Theorem 1 and its proof.

**Theorem 2.** If \(f \in R\), \(\lim_{z \to 0} |f(z)|(1 - |z|^2) < 1\) and \(L(f)\) is an infinite set, then there is a simple closed curve \(\gamma\) and points \(z_1, \ldots, z_j \neq 0\), such that \(L(f) = \gamma \cup \{z_1, \ldots, z_j\}\).

The points \(z_1, \ldots, z_j\) lie in the simply-connected bounded domain \(G\) defined by \(\partial G = \gamma\), \(G \subset D\). There are real numbers \(\alpha_{\nu,\mu}, \nu, \mu = 0, \ldots, 2r\), such that

\[
\sum_{\nu,\mu=0}^{2r} \alpha_{\nu,\mu} x^\nu y^\mu = 0 \quad \text{for any } z = x + iy \in L(f).
\]

**Proof.** According to Theorem 1 \(L(f)\) contains at least one simple closed curve \(\gamma\) which may be oriented such that \(n(\gamma, z) = 1\) for \(z \in G\). If we define \(h\) by (2) we get from (3),

\[(8) \quad 0 < |G| = \frac{1}{2\pi} \int_{\gamma} \bar{z} \, dz = \frac{1}{2\pi} \int_{\gamma} h(z) \, dz = \pi \sum_{z \in G} \text{Res } h(z).
\]

This means that \(h\) has \(r\) poles in \(G\), where \(r \geq 1\) and the poles are counted according to their order.

We now consider the Riemann surface \(h(G)\).

The boundary points of \(h(G)\) are the points of

\(\gamma^* := \{z \mid \bar{z} \in \gamma\} = h(\gamma)\).

Let \(G^*\) be defined by \(\partial G^* = \gamma^*, G^* \subset D\). Then for \(a \in G^*\) we get \(n(h(\gamma), a) = -1\) and for \(b \in \mathring{C} \setminus G^*\) \(n(h(\gamma), b) = 0\). It is a consequence of the argument principle that \(h\) takes the values \(a \in G^* \quad r - 1\) times and the values \(b \in \mathring{C} \setminus G^*\) \(r\) times in \(G\). Using continuity arguments it is easily seen that \(h\) takes any value \(c \in \gamma^* \quad r - 1\) times in \(G\) and \(r\) times in \(\mathring{G}\).
On the Riemann surface \( h(G) \) the inverse function of \( h \), \( h^{-1} \), is defined. The analytic continuation of \( h^{-1} \) across \( \gamma^* \) is delivered by (4) which yields the continuation
\[
h^{-1}(z) = \overline{h(\bar{z})} =: g(z)
\]
into \( \overline{G^*} \). \( g \) is meromorphic in \( G^* \), \( g(G^*) \cup G \) forms the Riemann surface of \( h \) wherein the analytic continuation of \( h \) across \( \gamma \) is defined by \( h = g^{-1} \). For later use we mention
\[
h(D \setminus \overline{G}) \subset G^*.
\]
The above considerations together with standard arguments for algebraic functions (compare for instance [1, V. 4]) show that \( h \) as well as \( h^{-1} \) are algebraic functions of order \( r \). Therefore we get an irreducible algebraic equation
\[
\sum_{k=0}^{r} \sum_{l=0}^{t} b_{kl} h^k z^l = 0, \quad b_{kl} \in \mathbb{C},
\]
to determine the analytic continuations of \( h \).

(4) and (11) show that
\[
\sum_{k=0}^{r} \sum_{l=0}^{t} b_{kl} \bar{z}^k \bar{h}^l = 0.
\]
Because of the irreducibility of (11) this implies the existence of \( \sigma \in \mathbb{C} \setminus \{0\} \) such that for any pair \( (k, l) \), \( 0 \leq k \leq r, 0 \leq l \leq t \),
\[
b_{kl} = \sigma b_{lk}.
\]
Hence \( b_{kl} = \sigma \bar{a}_{kl} \), \( \sigma = e^{2i\pi} \) and \( r = t \). With \( a_{kl} := e^{-r} b_{kl} \) we get instead of (10) and (12)
\[
\sum_{k=0}^{r} \sum_{l=0}^{r} a_{kl} h^k z^l = 0, \quad a_{kl} = \bar{a}_{lk}.
\]
According to (3) we have
\[
\gamma \subset L(f) \subset \left\{ z \mid \sum_{k,l=0}^{r} a_{kl} \bar{z}^k z^l = 0 \right\}.
\]
This proves the last assertion of Theorem 2 if we identify
\[
\sum_{k,l=0}^{r} a_{kl}(x - iy)^k(x + iy)^l = \sum_{\mu=0}^{2r} \alpha_{\mu} x^\mu y^\mu, \quad \alpha_{\mu} \in \mathbb{R}.
\]
The proof of Theorem 1 shows that for the function \( h \) defined in \( D \) by (2) there are, in a certain neighbourhood of \( \gamma \), no points \( z \) for which \( h(z) = \bar{z} \) except the points of \( \gamma \). Assuming the existence of a second closed simple curve \( \gamma' \subset L(f) \), which defines a bounded domain \( G' \) by \( \partial G' = \gamma' \), \( G' \subset D \), we therefore get three possibilities: (1) \( G' \subset D \setminus \overline{G} \), (2) \( \gamma \subset G' \), (3) \( \gamma' \subset G \).

In case (1) \( |G'| > 0 \) implies according to (3) and (8) that \( h \) has at least one pole in \( G' \subset D \setminus \overline{G} \) which contradicts (10).
In case (2) the inequality $|G'| > |G|$ implies the existence of a pole of $h$ in $G' \setminus \overline{G}$ which leads to the same contradiction.

Case (3) may be treated as case (2) interchanging the roles of $\gamma$ and $\gamma'$.

It remains to show that the points $z_1, \ldots, z_t$ lie in $G$. This is an immediate consequence of (3) and (10) since $G^* = \{ z \mid \bar{z} \in G \}$.

**Remark.** Theorem 2 answers in part the question of Cima and Wogen in [2] whether $L(f)$ is a connected set.

After the completion of the manuscript we found that meromorphic functions which fulfill the condition $h(z) = \bar{z}$ on a curve are known as Schwarz functions (compare P. Davis, *The Schwarz function and its applications*, Carus Math. Monographs, no. 17, Math. Assoc. Amer., Washington, D.C., 1974).

Another method to show some parts of the proof of Theorem 2 is given in the paper of D. Aharonov and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. 30 (1976), 39–73.

In a forthcoming paper St. Ruscheweyh and the present author will give further details on the construction of functions described in Theorem 2.

**References**


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