PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

WEIGHTED NORM INEQUALITIES FOR THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

WO-SANG YOUNG[1]

ABSTRACT. A characterization is obtained for weight functions v for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbf{R}^n, w\,dx)$ to $L^p(\mathbf{R}^n, v\,dx)$ for some nontrivial w.

In this note we obtain a necessary and sufficient condition on weight functions $v \geqslant 0$ such that the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbf{R}^n, w\,dx)$ to $L^p(\mathbf{R}^n, v\,dx)$ for some $w < \infty$ a.e. This answers a question posed by B. Muckenhoupt in [3]. The problem of characterizing all weight functions $w \geqslant 0$ for which there are nontrivial v's was solved independently by J. L. Rubio de Francia [4] and L. Carleson and P. W. Jones [1].

Let M be the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|\,dy,$$

where $B(x, r)$ is the ball of radius r centered at x and $|B(x, r)|$ is its Lebesgue measure. Our result is as follows.

THEOREM. *Given $v \geqslant 0$ and $1 < p < \infty$, the following conditions are equivalent*:

(a) *There is $w < \infty$ a.e. such that*

$$\int_{\mathbf{R}^n} |Mf|^p v\,dx \leqslant C \int_{\mathbf{R}^n} |f|^p w\,dx$$

for all $f \in L^p(\mathbf{R}^n, w\,dx)$.

(b)

$$\int_{\mathbf{R}^n} \frac{v(x)}{(1 + |x|^n)^p}\,dx < \infty.$$

In this paper C denotes a constant depending only on n and p, and may vary from line to line.

In [4] Rubio de Francia observed that (a) implies

$$(*) \qquad u \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \quad \text{and} \quad \left(\int_{|x| \leqslant R} u\right)^{1/p} = O(R^n) \qquad (R \to \infty),$$

Received by the editors September 1, 1981.

1980 *Mathematics Subject Classification*. Primary 42B25.

[1] Research supported in part by Natural Sciences and Engineering Research Council Canada Grant No. A5165.

and conjectured that it is also sufficient for (a). This is not the case since $u(x) = |x|^{n(p-1)}$ satisfies $(*)$ but not (b).

PROOF OF THEOREM. We first show that (a) implies (b). Since $w \not\equiv \infty$, there is a set A with positive measure in which w is bounded. Moreover, we can assume $A \subset \{|x| \leq R\}$ for some $1 < R < \infty$. Let $f = \chi_A$. Then

$$\int_{\mathbf{R}^n} |f|^p w\,dx = \int_A w < \infty.$$

For $|x| \leq R$, $Mf(x) \geq C|A|/R^n$, and for $|x| > R$, $Mf(x) \geq C|A|/|x|^n$. Hence

$$\int_{\mathbf{R}^n} |Mf|^p v\,dx \geq \left(C\frac{|A|}{R^n}\right)^p \int_{\mathbf{R}^n} \frac{v(x)}{(1+|x|^n)^p}\,dx.$$

Therefore

$$\int_{\mathbf{R}^n} \frac{v(x)}{(1+|x|^n)^p}\,dx < \infty.$$

We now prove that (b) implies (a). Let $v_1(x) = v(x)$ if $v(x) \geq 1$, and $v_1(x) = 1$ if $v(x) < 1$. Then v_1 also satisfies (b) because

$$\int_{\mathbf{R}^n} \frac{v_1(x)}{(1+|x|^n)^p}\,dx \leq \int_{\mathbf{R}^n} \frac{v(x)+1}{(1+|x|^n)^p}\,dx < \infty.$$

Since $v \leq v_1$, (a) is satisfied if we show that there is $w < \infty$ a.e. such that

$$\int_{\mathbf{R}^n} |Mf|^p v_1\,dx \leq C\int_{\mathbf{R}^n} |f|^p w\,dx \tag{1}$$

for all $f \in L^p(\mathbf{R}^n, w\,dx)$.

Let $u(x) = (1+|x|^n)^{1-p}$. We observe that $M(uv_1) < \infty$ a.e. To see this let $1 < R < \infty$ and consider $\{|x| \leq R\}$. For any such x,

$$\sup_{r \leq R} \frac{1}{r^n}\int_{|y-x|\leq r} uv_1 \leq CM(v_1\chi_{\{|y|\leq 2R\}})(x).$$

$M(v_1\chi_{\{|y|\leq 2R\}})$ is finite a.e. because $v_1 \in L^1_{\text{loc}}(\mathbf{R}^n)$. Also, for $r > R$

$$\frac{1}{r^n}\int_{|y-x|\leq r} uv_1 \leq \frac{1}{r^n}\int_{|y|\leq 2r} \frac{v_1(y)}{(1+|y|^n)^{p-1}}\,dy \leq C\int_{\mathbf{R}^n} \frac{v_1(y)}{(1+|y|^n)^p}\,dy < \infty.$$

Hence $M(uv_1) < \infty$ a.e.

Let $w = u^{-3}M(uv_1)$. Then $w < \infty$ a.e. We shall show that (1) holds for this w. Let $f \in L^p(\mathbf{R}^n, w\,dx)$, and, for $k = 0, 1, 2, \ldots$, let $f_k = f\chi_{\{2^k \leq |x| < 2^{k+1}\}}$. Then it follows from Fefferman and Stein [2] that

$$\begin{aligned}\int_{|x|\leq 2^{k+2}} |Mf_k|^p v_1 &\leq C2^{kn(p-1)}\int_{\mathbf{R}^n} |Mf_k|^p uv_1\\ &\leq C2^{kn(p-1)}\int_{\mathbf{R}^n} |f_k|^p M(uv_1)\\ &\leq C2^{-2kn(p-1)}\int_{\mathbf{R}^n} |f|^p w.\end{aligned}$$

For $|x| > 2^{k+2}$, we have

$$Mf_k(x) \leqslant \frac{C}{|x|^n} \int_{2^k \leqslant |x| < 2^{k+1}} |f|$$

$$\leqslant \frac{C}{|x|^n} \left(\int_{\mathbf{R}^n} |f|^p w \right)^{1/p} \left(\int_{2^k \leqslant |x| < 2^{k+1}} w^{-1/(p-1)} \right)^{1-1/p},$$

by Hölder's inequality. Therefore

$$\int_{|x|>2^{k+2}} |Mf_k|^p v_1 \leqslant C \left(\int_{\mathbf{R}^n} \frac{v_1(x)}{(1+|x|^n)^p} dx \right) \left(\int_{\mathbf{R}^n} |f|^p w \right) \left(\int_{2^k \leqslant |x| < 2^{k+1}} w^{-1/(p-1)} \right)^{p-1}.$$

To estimate the last factor we note that for $2^k \leqslant |x| < 2^{k+1}$,

$$M(uv_1)(x) \geqslant \frac{C}{2^{kn}} \int_{|y| \leqslant 2^k} \frac{v_1(y)}{(1+|y|^n)^{p-1}} dy \geqslant C 2^{-kn(p-1)},$$

since $v_1 \geqslant 1$. Hence, $w(x) \geqslant C 2^{2kn(p-1)}$, and

$$\left(\int_{2^k \leqslant |x| < 2^{k+1}} w^{-1/(p-1)} \right)^{p-1} \leqslant 2^{-kn(p-1)}.$$

Altogether we have

$$\int_{\mathbf{R}^n} |Mf_k|^p v_1 \leqslant C \left[\int_{\mathbf{R}^n} \frac{v(x)}{(1+|x|^n)^p} dx + 1 \right] \left(\int_{\mathbf{R}^n} |f|^p w \right) 2^{-kn(p-1)}.$$

It then follows from Minkowski's inequality that

$$\int_{\mathbf{R}^n} |Mf|^p v_1 \leqslant \left[\sum_{k=0}^{\infty} \left(\int_{\mathbf{R}^n} |Mf_k|^p v_1 \right)^{1/p} \right]^p$$

$$\leqslant C \left[\int_{\mathbf{R}^n} \frac{v(x)}{(1+|x|^n)^p} dx + 1 \right] \left(\int_{\mathbf{R}^n} |f|^p w \right) \left[\sum_{k=0}^{\infty} 2^{-kn(1-1/p)} \right]^p$$

$$\leqslant C \left[\int_{\mathbf{R}^n} \frac{v(x)}{(1+|x|^n)^p} dx + 1 \right] \left(\int_{\mathbf{R}^n} |f|^p w \right).$$

This complete the proof of the theorem.

I would like to thank R. A. Hunt, D. S. Kurtz and B. Muckenhoupt for their helpful conversations.

References

1. L. Carleson and P. W. Jones, *Weighted norm inequalities and a theorem of Koosis*, Mittag-Leffler Rep. No. 2, 1981.

2. C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

3. B. Muckenhoupt, *Weighted norm inequalities for classical operators*, Proc. Sympos. Pure Math., vol. 35, part 1, Amer. Math. Soc., Providence, R. I., 1979, pp. 69–83.

4. J. L. Rubio de Francia, *Boundedness of maximal functions and singular integrals in weighted L^p spaces*, Proc. Amer. Math. Soc. **83** (1981), 673–679.

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada