WEIGHTED NORM INEQUALITIES
FOR THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

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Abstract. A characterization is obtained for weight functions \(v\) for which the Hardy-Littlewood maximal operator is bounded from \(L^p(\mathbb{R}^n, wdx)\) to \(L^p(\mathbb{R}^n, vdx)\) for some nontrivial \(w\).

In this note we obtain a necessary and sufficient condition on weight functions \(v \geq 0\) such that the Hardy-Littlewood maximal operator is bounded from \(L^p(\mathbb{R}^n, wdx)\) to \(L^p(\mathbb{R}^n, vdx)\) for some \(w < \infty\) a.e. This answers a question posed by B. Muckenhoupt in [3]. The problem of characterizing all weight functions \(w \geq 0\) for which there are nontrivial \(v\)'s was solved independently by J. L. Rubio de Francia [4] and L. Carleson and P. W. Jones [1].

Let \(M\) be the Hardy-Littlewood maximal operator defined by

\[
Mf(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f| \, dy,
\]

where \(B(x,r)\) is the ball of radius \(r\) centered at \(x\) and \(|B(x,r)|\) is its Lebesgue measure. Our result is as follows.

Theorem. Given \(v \geq 0\) and \(1 < p < \infty\), the following conditions are equivalent:

(a) There is \(w < \infty\) a.e. such that

\[
\int_{\mathbb{R}^n} |Mf|^p v \, dx \leq C \int_{\mathbb{R}^n} |f|^p w \, dx
\]

for all \(f \in L^p(\mathbb{R}^n, wdx)\).

(b) \[
\int_{\mathbb{R}^n} \frac{v(x)}{(1 + |x|^p)^p} \, dx < \infty.
\]

In this paper \(C\) denotes a constant depending only on \(n\) and \(p\), and may vary from line to line.

In [4] Rubio de Francia observed that (a) implies

(*) \[
\forall f \in L^1_{\text{loc}}(\mathbb{R}^n) \quad \text{and} \quad \left( \int_{|x| \leq R} u \right)^{1/p} = O(R^n) \quad (R \to \infty),
\]

Received by the editors September 1, 1981.
1980 Mathematics Subject Classification. Primary 42B25.
\(^1\) Research supported in part by Natural Sciences and Engineering Research Council Canada Grant No. A5165.
and conjectured that it is also sufficient for (a). This is not the case since \( u(x) = |x|^{n(p-1)} \) satisfies (\(*\)) but not (b).

**Proof of Theorem.** We first show that (a) implies (b). Since \( w \equiv \infty \), there is a set \( A \) with positive measure in which \( w \) is bounded. Moreover, we can assume \( A \subset \{|x| \leq R\} \) for some \( 1 < R < \infty \). Let \( f = \chi_A \). Then

\[
\int_{\mathbb{R}^n} |f|^p w \, dx = \int_A w < \infty.
\]

For \( |x| \leq R \), \( Mf(x) \geq C |A| / R^n \), and for \( |x| > R \), \( Mf(x) \geq C |A| / |x|^n \). Hence

\[
\int_{\mathbb{R}^n} |Mf|^p v \, dx \geq \left( C \frac{|A|}{R^n} \right) \int_{\mathbb{R}^n} \frac{v(x)}{(1 + |x|^n)^p} \, dx.
\]

Therefore

\[
\int_{\mathbb{R}^n} \frac{v(x)}{(1 + |x|^n)^p} \, dx < \infty.
\]

We now prove that (b) implies (a). Let \( v_1(x) = v(x) \) if \( v(x) \geq 1 \), and \( v_1(x) = 1 \) if \( v(x) < 1 \). Then \( v_1 \) also satisfies (b) because

\[
\int_{\mathbb{R}^n} \frac{v_1(x)}{(1 + |x|^n)^p} \, dx \leq \int_{\mathbb{R}^n} \frac{v(x) + 1}{(1 + |x|^n)^p} \, dx < \infty.
\]

Since \( v \leq v_1 \), (a) is satisfied if we show that there is \( w < \infty \) a.e. such that

(1) \[
\int_{\mathbb{R}^n} |Mf|^p v_1 \, dx \leq C \int_{\mathbb{R}^n} |f|^p w \, dx
\]

for all \( f \in L^p(\mathbb{R}^n, w \, dx) \).

Let \( u(x) = (1 + |x|^n)^{1-p} \). We observe that \( M(uv_1) < \infty \) a.e. To see this let \( 1 < R < \infty \) and consider \( \{|x| \leq R\} \). For any such \( x \),

\[
\sup_{r \leq R} \frac{1}{r^n} \int_{|y-x|<r} uv_1 \leq CM(v_1 \chi_{|y|<2R})(x).
\]

\( M(v_1 \chi_{|y|<2R}) \) is finite a.e. because \( v_1 \in L^1_{\text{loc}}(\mathbb{R}^n) \). Also, for \( r > R \)

\[
\frac{1}{r^n} \int_{|y-x|<r} uv_1 \leq \frac{1}{r^n} \int_{|y|<2r} \frac{v_1(y)}{(1 + |y|^n)^{p-1}} \, dy \leq C \int_{\mathbb{R}^n} \frac{v_1(y)}{(1 + |y|^n)^p} \, dy < \infty.
\]

Hence \( M(uv_1) < \infty \) a.e.

Let \( w = u^{-2}M(uv_1) \). Then \( w < \infty \) a.e. We shall show that (1) holds for this \( w \). Let \( f \in L^p(\mathbb{R}^n, w \, dx) \), and, for \( k = 0, 1, 2, \ldots \), let \( f_k = f \chi_{\{|x| \leq 2^{k+1}\}} \). Then it follows from Fefferman and Stein [2] that

\[
\int_{|x| \leq 2^{k+2}} |Mf_k|^p v_1 \leq C 2^{kn(p-1)} \int_{\mathbb{R}^n} |Mf_k|^p w v_1
\]

\[
\leq C 2^{kn(p-1)} \int_{\mathbb{R}^n} |f_k|^p w M(uv_1)
\]

\[
\leq C 2^{-kn(p-1)} \int_{\mathbb{R}^n} |f|^p w.
\]
For $|x| > 2^{k+2}$, we have
\[
Mf_k(x) \leq \frac{C}{|x|^p} \int_{2^k \leq |x| < 2^{k+1}} |f|
\]
\[
\leq \frac{C}{|x|^p} \left( \int_{\mathbb{R}^n} |f|^p w \right)^{1/p} \left( \int_{2^k \leq |x| < 2^{k+1}} w^{-1/(p-1)} \right)^{1-1/p},
\]
by Hölder's inequality. Therefore
\[
\int_{|x| > 2^{k+2}} |Mf_k|^p v_1 \leq C \left( \int_{\mathbb{R}^n} \frac{v_1(x)}{1 + |x|^p} \right) \left( \int_{\mathbb{R}^n} |f|^p w \right) \left( \int_{2^k \leq |x| < 2^{k+1}} w^{-1/(p-1)} \right)^{p-1}.
\]
To estimate the last factor we note that for $2^k \leq |x| < 2^{k+1}$,
\[
M(w_1)(x) \geq \frac{C}{2^{kn}} \int_{|y| \leq 2^k} \frac{v_1(y)}{1 + |y|^p} dy \geq C2^{-kn(p-1)},
\]
since $v_1 \geq 1$. Hence, $w(x) \geq C2^{kn(p-1)}$, and
\[
\left( \int_{2^k \leq |x| < 2^{k+1}} w^{-1/(p-1)} \right)^{p-1} \leq 2^{-kn(p-1)}.
\]
Altogether we have
\[
\int_{\mathbb{R}^n} |Mf_k|^p v_1 \leq C \left[ \int_{\mathbb{R}^n} \frac{v(x)}{1 + |x|^p} dx + 1 \right] \left( \int_{\mathbb{R}^n} |f|^p w \right) 2^{-kn(p-1)}.
\]
It then follows from Minkowski's inequality that
\[
\int_{\mathbb{R}^n} |Mf|^p v_1 \leq \left[ \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^n} |Mf_k|^p v_1 \right)^{1/p} \right]^p
\]
\[
\leq C \left[ \int_{\mathbb{R}^n} \frac{v(x)}{1 + |x|^p} dx + 1 \right] \left( \int_{\mathbb{R}^n} |f|^p w \right) \left[ \sum_{k=0}^{\infty} 2^{-kn(1-1/p)} \right]^p
\]
\[
\leq C \left[ \int_{\mathbb{R}^n} \frac{v(x)}{1 + |x|^p} dx + 1 \right] \left( \int_{\mathbb{R}^n} |f|^p w \right).
\]
This complete the proof of the theorem.
I would like to thank R. A. Hunt, D. S. Kurtz and B. Muckenhoupt for their helpful conversations.

REFERENCES

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