QUOTIENTS OF BANACH SPACES OF COTYPE q

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Abstract. Let Z be a Banach space and let X \subset Z be a B-convex subspace (equivalently, assume that X does not contain l_1^n's uniformly). Then every Bernoulli series \( \sum_{n=1}^{\infty} e_n x_n \) which converges almost surely in the quotient Z/X can be lifted to a Bernoulli series a.s. convergent in Z. As a corollary, if Z is of cotype q, then Z/X is also of cotype q. This extends a result of [4] concerning the particular case Z = L_1.

In this note, we give some further applications of the main results of [5]. It is well known that, in general, an unconditionally convergent series in a quotient Banach space Z/X cannot be lifted up to an unconditionally convergent series in Z. However, we prove in this note that if X is B-convex, then a similar lifting property holds for "almost" unconditionally convergent series. (Of course, this becomes trivial if the subspace X is complemented in Z.)

We will need some specific notations: let D = \{-1, 1\}^N, e_n: D \to \{-1, +1\} the nth coordinate, and let \( \mu \) denote the uniform probability on D (i.e. \( \mu \) is the normalized Haar measure on the compact group D). For any finite set of integers A \subset \mathbb{N}, we denote by \( W_A \) the Walsh function \( W_A = \prod_{n \in A} e_n \). These functions form an orthonormal basis of characters of the space \( L_2(D, \mu) \).

We will denote by \( R_k \) the orthogonal projection from \( L_2(D, \mu) \) onto the closed linear span of the functions \( \{ W_A \mid |A| = k \} \); moreover, we will denote by \( \mathcal{P} \) the linear span of all the functions of the form \( W_A \) (i.e. \( \mathcal{P} \) is the space of all "trigonometric polynomials" on the group D). It is well known that, for each \( \varepsilon \) in \([0, 1]\), the operator \( T(\varepsilon) = \sum_{k \geq 0} \varepsilon^k R_k \) (defined a priori only on \( \mathcal{P} \)) extends to a contraction on \( L_p(D, \mu) \) for each \( p \) such that \( 1 < p < \infty \).

Now let Z be an arbitrary Banach space. We denote by \( I_Z \) the identity operator on Z. Obviously the operator \( T(\varepsilon) \otimes I_Z \) (defined a priori only on \( \mathcal{P} \otimes Z \)) extends to a linear contraction—which we still denote \( T(\varepsilon) \otimes I_Z \)—on the space \( L_p(D, \mu; Z) \) for \( 1 < p < \infty \). For simplicity, we will write in the sequel \( L_p(Z) \) instead of \( L_p(D, \mu; Z) \). We will denote also \( B(L_p(Z)) \), the Banach space of all bounded operators on \( L_p(Z) \).

Now let X be a B-convex Banach space (equivalently a space which does not contain l_1^n's uniformly, see [3] for details). It was proved in [5] that there exists a constant \( C > 1 \) (depending only on X) such that the operator \( R_k \otimes I_X \) (defined a...
priori only on $\mathcal{H} \otimes X$) defines a bounded operator on $L_2(X)$ and verifies

$$\forall k > 0, \quad \|R_k \otimes I_X\|_{L_2(H(X))} \leq C^k.$$  

We can now show the main result of this note.

**Theorem.** Let $Z$ be an arbitrary Banach space and let $X \subset Z$ be a $B$-convex closed subspace. Denote by $\pi: Z \to Z/X$ the canonical surjection. Then for any sequence $(y_n)$ in $Z/X$ such that the series $\sum_{n=1}^{\infty} \varepsilon_n y_n$ converges in $L^2(Z/X)$, we can find a sequence $(z_n)$ in $Z$, such that $\pi(z_n) = y_n$ for each $n$, and the series $\sum_{n=1}^{\infty} \varepsilon_n z_n$ converges in $L^2(Z)$ and verifies

$$\sum_{n=1}^{\infty} \varepsilon_n z_n \in L^2(Z)$$

for some constant $K$ depending only on $X$.

**Proof.** Obviously, it is enough to prove the theorem assuming that $(y_n)$ is a finite sequence $(y_1, \ldots, y_N)$ (so that $y_n = 0$ for all $n > N$). Denote by $P_N$ the set of all finite subsets of $\{1, 2, \ldots, N\}$.

Assume that $\|\sum_{n=1}^{N} \varepsilon_n y_n\|_{L^2(Z/X)} < 1$.

By an obvious pointwise lifting, we can find a function $\Phi: D \to Z$, depending only on the $N$ first coordinates, and such that $\|\Phi\|_{L^2(Z)} < 1$ and

$$\pi(\Phi) = \sum_{n=1}^{N} \varepsilon_n y_n.$$  

We may as well assume that $\Phi$ is an odd function on $D$, i.e. that $\Phi(\omega) = -\Phi(-\omega)$ $\forall \omega \in D$ (otherwise, we replace $\Phi(\omega)$ by $\frac{1}{2}(\Phi(\omega) - \Phi(-\omega))$).

A priori, $\Phi$ admits a development as follows:

$$\Phi = \sum_{A \in P_N} W_A z_A \quad \text{with } z_A \in Z.$$  

$(z_A$ is defined as $\int \Phi W_A \, d\mu)$.

The equality (3) implies that $\pi(z_n) = y_n$ $\forall n = 1, 2, \ldots, N$, and

$$\pi(z_A) = 0 \quad \text{if } |A| \neq 1.$$  

Therefore, $z_A$ belongs to $X$ whenever $|A| \neq 1$; moreover, since $\Phi$ is odd, we have $z_A = 0$ whenever $|A|$ is even.

We claim that the sequence $(z_{(1)}, \ldots, z_{(N)})$ verifies the desired property:

Let us write for simplicity,

$$\Phi_k = R_k \otimes I_{Z}(\Phi),$$

so that $\Phi = \sum_{k \geq 0; k \text{ odd}} \Phi_k$, and $\Phi_k = 0$ for all even $k$.

Since $T(\varepsilon) \otimes I_Z$ is a contraction on $L_2(Z)$, and since $T(\varepsilon) \otimes I_Z(\Phi) = \sum_{k \geq 0} \varepsilon^k \Phi_k$, we have

$$\left\|\sum_{k \geq 0} \varepsilon^k \Phi_k\right\|_{L^2(Z)} < 1.$$
It follows that

\[ \varepsilon \| \Phi_1 \|_{L_2(Z)} < 1 + \sum_{k \geq 3} \varepsilon^k \Phi_k \|_{L_2(Z)}. \]

But by (4) we know that \( \Phi_k \) actually belongs to \( L_2(X) \) for each \( k \geq 3 \), and we may as well assume that \( X \) verifies (1), so that we have, for each \( \varepsilon < 1/C \),

\[ \left\| \sum_{k \geq 3} \varepsilon^k \Phi_k \right\|_{L_2(X)} < \sum_{k \geq 3} \varepsilon^k C^k \| \Phi - \Phi_1 \|_{L_2(X)} \]

\[ < (\varepsilon C)^3 (1 - \varepsilon C)^{-1} \| \Phi - \Phi_1 \|_{L_2(X)} \]

\[ < (\varepsilon C)^3 (1 - \varepsilon C)^{-1} (1 + \| \Phi_1 \|_{L_2(Z)}). \]

Combining (5) and (6), we find

\[ \varepsilon \| \Phi_1 \|_{L_2(Z)} < 1 + (\varepsilon C)^3 (1 - \varepsilon C)^{-1} (1 + \| \Phi_1 \|_{L_2(Z)}). \]

Now, if we choose \( \varepsilon \) such that \( 2C(\varepsilon C)^2 = \frac{1}{2} \), i.e. \( \varepsilon = (4C^3)^{-1/2} \), we have \( (1 - \varepsilon C)^{-1} < 2 \) and (7) yields

\[ \| \Phi_1 \|_{L_2(Z)} < K, \]

with \( K = 2(1 + \frac{1}{2}(4C^3)^{-1/2}) (4C^3)^{1/2} \).

By homogeneity, this concludes the proof of (2) in the finite case, and hence completes the proof of the theorem.

A Banach space \( Z \) is called of cotype \( q \) if there is a constant \( \lambda \) such that, for any finite sequence \( (z_n) \) in \( Z \), we have

\[ \left( \sum \| z_n \|^q \right)^{1/q} < \lambda \left\| \sum \varepsilon_n z_n \right\|_{L_2(Z)}. \]

(See [3] for more details on this notion.)

It is well known that \( L_1 \)-spaces are of cotype 2. It is also well known that, in general, a quotient of a cotype \( q \) space need not be of cotype \( q \). However, it was proved in [4] that if \( R \) is a \( B \)-convex subspace of \( L_1 \), then \( L_1/R \) is of cotype 2. (Without any restriction on \( R \), this is certainly false since for example \( c_0 \) is isometric to a quotient of \( l_1 \).) The next corollary generalizes this last result.

**Corollary 1.** In the situation of the theorem, if \( Z \) is of cotype \( q \), then \( Z/X \) is also of cotype \( q \).

**Corollary 2.** Assume that \( X \subset Z \) does not contain \( l_1^n \)'s uniformly. If \( Z \) does not contain \( l_\infty^n \)'s uniformly, the same is true for \( Z/X \).

**Proof.** This follows from the theorem and Theorem 1.1 in [3].

It is natural to ask whether there exists a "geometric" proof of the last result. Also, there might be an infinite dimensional analogue concerning e.g. spaces which do not contain \( l_1 \). Moreover, we do not know, in the situation of the theorem, whether or not we can lift unconditionally convergent series in \( Z/X \) into unconditionally convergent series in \( Z \).

The reader will have noticed that, when \( X \) is a "concrete" Banach space, for instance when \( X \) is a Hilbert space, or when \( X \) embeds in \( L_p \) for some \( 1 < p < \infty \),
then the results of [5] are not needed in the proof of the above theorem. Therefore, we have also obtained a more direct proof of the result of [4], that $L_1/R$ is of cotype 2 whenever $R$ is a reflexive subspace of $L_1$. (Recall that, by [6], $R \subset L_1$ is reflexive iff $R$ is $B$-convex and in that case $R$ embeds in $L_p$ for some $p > 1$.)

**Remarks.** We mention here some easy generalizations of the previous results:

(i) By a result of Kahane (cf. [2], p. 17), for any $0 < p < \infty$ and any Banach space $Z$, a Bernoulli series $\sum_{n=1}^{\infty} e_n z_n$ is convergent in $L_p(Z)$ iff it converges a.s. in $Z$. Therefore, we may also state the theorem as it is in the abstract.

(ii) In the same situation as in the theorem, we can prove using the same basic idea: For each $k > 0$ there is a constant $K(k)$ such that, for every integer $N$, every set $\{y_A | A \in P_N, |A| = k\}$ in $Z/X$ can be lifted to a subset $\{z_A | A \in P_N, |A| = k\}$ of $Z$, which verifies

$$\left\| \sum_{A \in P_N} W_A z_A \right\|_{L_p(Z)} < K(k) \left\| \sum_{A \in P_N} W_A y_A \right\|_{L_p(Z/X)}.$$ 

Finally it is easy to generalize the proof of the above theorem as follows:

**Corollary 3.** In the situation of the theorem, let $(Y_n)$ be a sequence of independent random variables with values in $Z/X$. Fix $p$ such that $1 < p < \infty$. Assume that $\sum_{n=1}^{\infty} Y_n$ converges a.s. (resp. in $L_p(Z/X)$); then there exist a sequence $(Z_n)$ of independent $Z$-valued random variables, such that $Z_n$ is $Y_n$-measurable, $\sum_{n=1}^{\infty} Z_n$ converges a.s. (resp. in $L_p(Z)$), and $\pi(Z_n) = Y_n$.

In the preceding statement, we implicitly assume that all the random variables considered have a separable range.

**Proof.** The part concerning the convergence in $L_p(Z)$ can be proved exactly as in the theorem but using the proof of Corollary 3.4 instead of Theorem 2.1 in [5]. More precisely, let $\mathcal{E}_n$ be the $\sigma$-algebra generated by $\{Y_m | m \neq n\}$, and let $V_k$ be the projection defined on $L_p(d\mathcal{P})$ by

$$V_k = \sum_{A \subset N, |A| = k} \prod_{j \in A} (I - E_{e_j}) \prod_{j \notin A} E_{e_j}.$$ 

The proof of Corollary 3.4 in [5] shows that if $X$ is $B$-convex, then there exists a constant $C$ such that

$$\forall k > 1 \quad \|V_k \otimes I_X\|_{B(L_p(X))} < C^k.$$ 

The first part of the proof can then be completed by reasoning as we did to prove the theorem.

Now, if $\sum_{n=1}^{\infty} Y_n$ converges a.s. in $Z/X$, we define $Y'_n = Y_n 1_{\{\|Y_n\| < 1\}}$. By Corollary 3.3 in [1], the series $\sum_{n=1}^{\infty} Y'_n$ converges in $L_p(Z/X)$ for each $p < \infty$. Therefore by the first part of the proof we can find, for each $n$, a $Z$-valued variable $Z'_n$ which is $Y'_n$-measurable and such that $\sum_{n=1}^{\infty} Z'_n$ converges (say) in $L_2(Z)$ and $\pi(Z'_n) = Y'_n$. Since the variables $(Z'_n)$ are independent, $\sum_{n=1}^{\infty} Z'_n$ converges also a.s. By lifting trivially $Y_n - Y'_n$, we can find a $Y_n$-measurable $Z$-valued variable $Z''_n$ supported by the set $\{\|Y_n\| > 1\}$ and such that $\pi(Z''_n) = Y_n - Y'_n$. 
Clearly, if we define $Z_n = Z'_n + Z''_n$, the series $\sum_{n=1}^{\infty} Z_n$ converges a.s. in $\mathbb{Z}$ and $\pi(Z_n) = Y_n$ for each $n$. Q.E.D.

Remark. Let $(g_n)_{n \geq 1}$ be a sequence of independent, equidistributed, standard Gaussian variables. Using Remark 2.2 in [5], it is easy to prove the above theorem with the sequence $(g_n)$ instead of $(\varepsilon_n)$. By known results on Gaussian measures, this yields immediately: In the situation of the theorem, any Gaussian Radon measure on $\mathbb{Z}/X$ is the image (by the canonical surjection) of a Gaussian Radon measure on $\mathbb{Z}$.

REFERENCES


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