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QUOTIENTS OF BANACH SPACES OF COTYPE q

GILLES PISIER

ABSTRACT. Let Z be a Banach space and let $X \subset Z$ be a B-convex subspace (equivalently, assume that X does not contain l_1^n's uniformly). Then every Bernoulli series $\sum_{n=1}^{\infty} \varepsilon_n z_n$ which converges almost surely in the quotient Z/X can be lifted to a Bernoulli series a.s. convergent in Z. As a corollary, if Z is of cotype q, then Z/X is also of cotype q. This extends a result of [4] concerning the particular case $Z = L_1$.

In this note, we give some further applications of the main results of [5]. It is well known that, in general, an unconditionally convergent series in a quotient Banach space Z/X cannot be lifted up to an unconditionally convergent series in Z. However, we prove in this note that if X is B-convex, then a similar lifting property holds for "almost" unconditionally convergent series. (Of course, this becomes trivial if the subspace X is complemented in Z.)

We will need some specific notations: let $D = \{-1, 1\}^{\mathbf{N}}$, $\varepsilon_n: D \to \{-1, +1\}$ the nth coordinate, and let μ denote the uniform probability on D (i.e. μ is the normalized Haar measure on the compact group D). For any finite set of integers $A \subset \mathbf{N}$, we denote by W_A the Walsh function $W_A = \prod_{n \in A} \varepsilon_n$. These functions form an orthonormal basis of characters of the space $L_2(D, \mu)$.

We will denote by R_k the orthogonal projection from $L_2(D, \mu)$ onto the closed linear span of the functions $\{W_A \mid |A| = k\}$; moreover, we will denote by $\mathcal{P}$ the *linear* span of all the functions of the form W_A (i.e. $\mathcal{P}$ is the space of all "trigonometric polynomials" on the group D). It is well known that, for each ε in $[0, 1]$, the operator $T(\varepsilon) = \sum_{k \geq 0} \varepsilon^k R_k$ (defined a priori only on $\mathcal{P}$) extends to a contraction on $L_p(D, \mu)$ for each p such that $1 < p < \infty$.

Now let Z be an arbitrary Banach space. We denote by I_Z the identity operator on Z. Obviously the operator $T(\varepsilon) \otimes I_Z$ (defined a priori only on $\mathcal{P} \otimes Z$) extends to a linear contraction—which we still denote $T(\varepsilon) \otimes I_Z$—on the space $L_p(D, \mu; Z)$ for $1 < p < \infty$. For simplicity, we will write in the sequel $L_p(Z)$ instead of $L_p(D, \mu; Z)$. We will denote also $B(L_p(Z))$, the Banach space of all bounded operators on $L_p(Z)$.

Now let X be a B-convex Banach space (equivalently a space which does not contain l_1^n's uniformly, see [3] for details). It was proved in [5] that there exists a constant $C \geq 1$ (depending only on X) such that the operator $R_k \otimes I_X$ (defined a

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priori only on $\mathcal{P} \otimes X$) defines a bounded operator on $L_2(X)$ and verifies

(1) $$\forall k \geqslant 0, \qquad \|R_k \otimes I_X\|_{B(L_2(X))} \leqslant C^k.$$

We can now show the main result of this note.

THEOREM. *Let Z be an arbitrary Banach space and let $X \subset Z$ be a B-convex closed subspace. Denote by π: $Z \to Z/X$ the canonical surjection. Then for any sequence (y_n) in Z/X such that the series $\sum_{n=1}^{\infty} \varepsilon_n y_n$ converges in $L_2(Z/X)$, we can find a sequence (z_n) in Z, such that $\pi(z_n) = y_n$ for each n, and the series $\sum_{n=1}^{\infty} \varepsilon_n z_n$ converges in $L_2(Z)$ and verifies*

(2) $$\left\| \sum_{n=1}^{\infty} \varepsilon_n z_n \right\|_{L_2(Z)} \leqslant K \left\| \sum_{n=1}^{\infty} \varepsilon_n y_n \right\|_{L_2(Z/X)},$$

for some constant K depending only on X.

PROOF. Obviously, it is enough to prove the theorem assuming that (y_n) is a finite sequence $(y_1, \ldots, y_N)$ (so that $y_n = 0$ for all $n > N$). Denote by P_N the set of all finite subsets of $\{1, 2, \ldots, N\}$.

Assume that $\|\sum_{n=1}^{n=N} \varepsilon_n y_n\|_{L_2(Z/X)} < 1$.

By an obvious pointwise lifting, we can find a function Φ: $D \to Z$, depending only on the N first coordinates, and such that $\|\Phi\|_{L_2(Z)} < 1$ and

(3) $$\pi(\Phi) = \sum_{n=1}^{N} \varepsilon_n y_n.$$

We may as well assume that Φ is an odd function on D, i.e. that $\Phi(\omega) = -\Phi(-\omega)$ $\forall \omega \in D$ (otherwise, we replace $\Phi(\omega)$ by $\frac{1}{2}(\Phi(\omega) - \Phi(-\omega))$.

A priori, Φ admits a development as follows:

$$\Phi = \sum_{A \in P_N} W_A z_A \quad \text{with } z_A \in Z.$$

(z_A is defined as $\int \Phi W_A \, d\mu$.)

The equality (3) implies that $\pi(z_{\{n\}}) = y_n \ \forall n = 1, 2, \ldots, N$, and

(4) $$\pi(z_A) = 0 \quad \text{if } |A| \neq 1.$$

Therefore, z_A belongs to X whenever $|A| \neq 1$; moreover, since Φ is odd, we have $z_A = 0$ whenever $|A|$ is even.

We claim that the sequence $\{z_{\{1\}}, \ldots, z_{\{N\}}\}$ verifies the desired property:

Let us write for simplicity,

$$\Phi_k = R_k \otimes I_Z(\Phi),$$

so that $\Phi = \sum_{k>0;\, k \text{ odd}} \Phi_k$, and $\Phi_k = 0$ for all even k.

Since $T(\varepsilon) \otimes I_Z$ is a contraction on $L_2(Z)$, and since $T(\varepsilon) \otimes I_Z(\Phi) = \sum_{k>0} \varepsilon^k \Phi_k$, we have

$$\left\| \sum_{k>0} \varepsilon^k \Phi_k \right\|_{L_2(Z)} < 1.$$

It follows that

$$\varepsilon\|\Phi_1\|_{L_2(Z)} \leqslant 1 + \Big\| \sum_{k \geqslant 3} \varepsilon^k \Phi_k \Big\|_{L_2(Z)}. \tag{5}$$

But by (4) we know that Φ_k actually belongs to $L_2(X)$ for each $k \geqslant 3$, and we may as well assume that X verifies (1), so that we have, for each $\varepsilon < 1/C$,

$$\begin{aligned} \Big\| \sum_{k \geqslant 3} \varepsilon^k \Phi_k \Big\|_{L_2(X)} &\leqslant \sum_{k \geqslant 3} \varepsilon^k C^k \|\Phi - \Phi_1\|_{L_2(X)} \\ &\leqslant (\varepsilon C)^3 (1 - \varepsilon C)^{-1} \|\Phi - \Phi_1\|_{L_2(X)} \\ &\leqslant (\varepsilon C)^3 (1 - \varepsilon C)^{-1} (1 + \|\Phi_1\|_{L_2(Z)}). \end{aligned} \tag{6}$$

Combining (5) and (6), we find

$$\varepsilon\|\Phi_1\|_{L_2(Z)} \leqslant 1 + (\varepsilon C)^3 (1 - \varepsilon C)^{-1} (1 + \|\Phi_1\|_{L_2(Z)}). \tag{7}$$

Now, if we choose ε such that $2C(\varepsilon C)^2 = \frac{1}{2}$, i.e. $\varepsilon = (4C^3)^{-1/2}$, we have $(1 - \varepsilon C)^{-1} \leqslant 2$ and (7) yields

$$\|\Phi_1\|_{L_2(Z)} \leqslant K, \tag{8}$$

with $K = 2\{1 + \frac{1}{2}(4C^3)^{-1/2}\}(4C^3)^{1/2}$.

By homogeneity, this concludes the proof of (2) in the finite case, and hence completes the proof of the theorem.

A Banach space Z is called of cotype q if there is a constant λ such that, for any finite sequence (z_n) in Z, we have

$$\Big(\sum \|z_n\|^q\Big)^{1/q} \leqslant \lambda \Big\| \sum \varepsilon_n z_n \Big\|_{L_2(Z)}.$$

(See [3] for more details on this notion.)

It is well known that L_1-spaces are of cotype 2. It is also well known that, in general, a quotient of a cotype q space need not be of cotype q. However, it was proved in [4] that if R is a B-convex subspace of L_1, then L_1/R is of cotype 2. (Without any restriction on R, this is certainly false since for example c_0 is isometric to a quotient of l_1.) The next corollary generalizes this last result.

COROLLARY 1. *In the situation of the theorem, if Z is of cotype q, then Z/X is also of cotype q.*

COROLLARY 2. *Assume that $X \subset Z$ does not contain l_1^n's uniformly. If Z does not contain l_∞^n's uniformly, the same is true for Z/X.*

PROOF. This follows from the theorem and Theorem 1.1 in [3].

It is natural to ask whether there exists a "geometric" proof of the last result. Also, there might be an infinite dimensional analogue concerning e.g. spaces which do not contain l_1. Moreover, we do not know, in the situation of the theorem, whether or not we can lift unconditionally convergent series in Z/X into unconditionally convergent series in Z.

The reader will have noticed that, when X is a "concrete" Banach space, for instance when X is a Hilbert space, or when X embeds in L_p for some $1 < p < \infty$, then the results of [5] are not needed in the proof of the above theorem. Therefore, we have also obtained a more direct proof of the result of [4], that L_1/R is of cotype 2 whenever R is a reflexive subspace of L_1. (Recall that, by [6], $R \subset L_1$ is reflexive iff R is B-convex and in that case R embeds in L_p for some $p > 1$.)

REMARKS. We mention here some easy generalizations of the previous results:

(i) By a result of Kahane (cf. [2], p. 17), for any $0 < p < \infty$ and any Banach space Z, a Bernoulli series $\sum_{n=1}^{\infty} \varepsilon_n z_n$ is convergent in $L_p(Z)$ iff it converges a.s. in Z. Therefore, we may also state the theorem as it is in the abstract.

(ii) In the same situation as in the theorem, we can prove using the same basic idea: For each $k > 0$ there is a constant $K(k)$ such that, for every integer N, every set $\{y_A | A \in P_N, |A| = k\}$ in Z/X can be lifted to a subset $\{z_A | A \in P_N, |A| = k\}$ of Z, which verifies

$$\Big\| \sum_{\substack{A \in P_N \\ |A| = k}} W_A z_A \Big\|_{L_2(Z)} \le K(k) \Big\| \sum_{\substack{A \in P_N \\ |A| = k}} W_A y_A \Big\|_{L_2(Z/X)}.$$

Finally it is easy to generalize the proof of the above theorem as follows:

COROLLARY 3. *In the situation of the theorem, let (Y_n) be a sequence of independent random variables with values in Z/X. Fix p such that $1 < p < \infty$. Assume that $\sum_{n=1}^{\infty} Y_n$ converges a.s. (resp. in $L_p(Z/X)$); then there exist a sequence (Z_n) of independent Z-valued random variables, such that Z_n is Y_n-measurable, $\sum_{n=1}^{\infty} Z_n$ converges a.s. (resp. in $L_p(Z)$), and $\pi(Z_n) = Y_n$.*

In the preceding statement, we implicitly assume that all the random variables considered have a separable range.

PROOF. The part concerning the convergence in $L_p(Z)$ can be proved exactly as in the theorem but using the proof of Corollary 3.4 instead of Theorem 2.1 in [5]. More precisely, let $\mathcal{Q}_n'$ be the σ-algebra generated by $\{Y_m | m \neq n\}$, and let V_k be the projection defined on $L_p(d\mathbf{P})$ by

$$V_k = \sum_{\substack{A \subset \mathbf{N} \\ |A| = k}} \prod_{j \in A} (I - \mathbf{E}^{\mathcal{Q}_j'}) \prod_{j \notin A} \mathbf{E}^{\mathcal{Q}_j'}.$$

The proof of Corollary 3.4 in [5] shows that if X is B-convex, then there exists a constant C such that

$$\forall k \geq 1 \quad \| V_k \otimes I_X \|_{B(L_p(X))} \le C^k.$$

The first part of the proof can then be completed by reasoning as we did to prove the theorem.

Now, if $\sum_{n=1}^{\infty} Y_n$ converges a.s. in Z/X, we define $Y_n' = Y_n 1_{\{\|Y_n\| < 1\}}$. By Corollary 3.3 in [1], the series $\sum_{n=1}^{\infty} Y_n'$ converges in $L_p(Z/X)$ for each $p < \infty$. Therefore by the first part of the proof we can find, for each n, a Z-valued variable Z_n' which is Y_n'-measurable and such that $\sum_{n=1}^{\infty} Z_n'$ converges (say) in $L_2(Z)$ and $\pi(Z_n') = Y_n'$. Since the variables (Z_n') are independent, $\sum_{n=1}^{\infty} Z_n'$ converges also a.s. By lifting trivially $Y_n - Y_n'$, we can find a Y_n-measurable Z-valued variable Z_n'' supported by the set $\{\|Y_n\| \geq 1\}$ and such that $\pi(Z_n'') = Y_n - Y_n'$.

Clearly, if we define $Z_n = Z_n' + Z_n''$, the series $\sum_{n=1}^{\infty} Z_n$ converges a.s. in Z and $\pi(Z_n) = Y_n$ for each n. Q.E.D.

REMARK. Let $(g_n)_{n \geq 1}$ be a sequence of independent, equidistributed, standard Gaussian variables. Using Remark 2.2 in [5], it is easy to prove the above theorem with the sequence (g_n) instead of (ε_n). By known results on Gaussian measures, this yields immediately: In the situation of the theorem, any Gaussian Radon measure on Z/X is the image (by the canonical surjection) of a Gaussian Radon measure on Z.

REFERENCES

1. J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math. **52** (1974), 159–186.

2. J. P, Kahane, *Some random series of functions*, Heath Math. Monograph, Raytheon Education Company, Lexington, Mass., 1968.

3. B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. **58** (1976), 45–90.

4. G. Pisier, *Une nouvelle class d'espaces vérifiant le théorème de Grothendieck*, Ann. Inst. Fourier (Grenoble) **28** (1978), 69–90.

5. ______, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. of Math. (to appear).

6. H. P. Rosenthal, *On subspaces of L_1*, Ann. of Math. (2) **97** (1973), 344–373.

CENTRE DE MATHÉMATIQUES DE L'ÉCOLE POLYTECHNIQUE, PLATEAU DE PALAISEAU, 91128 PALAISEAU CEDEX, FRANCE

Current address: Equipe d'Analyse, Université Paris 6, Tour 46, 4[e] Etage, 4 Place Jussieu, 75230 Paris, Cedex 05, France