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A FIXED POINT THEOREM FOR THE SUM OF TWO MAPPINGS

OLGA HADŽIĆ

ABSTRACT. A generalization of a fixed point theorem of Rzepecki is proved and it is shown that in a paranormed space E this result yields, under certain circumstances, solutions to the equation $x = Tx + Sx$ for $T: E \to E$ either continuous and affine or a generalized contraction, and $S: K \subseteq E \to E$ compact.

In [7] Zima proved a generalization of the Schauder fixed point theorem in a paranormed space setting. (Paranormed spaces are nonlocally convex topological vector spaces; see the definition below.) B. Rzepecki then proved the following generalization of Zima's result.

THEOREM 1 [6]. *Let X be a Hausdorff topological vector space, K be a nonempty, closed and convex subset of X and T be a continuous mapping from K into a compact set Z ($Z \subset K$). Suppose that for every $x \in Z$ and every neighborhood V of x there exists a neighborhood U of x such that*

$$\operatorname{co}(U \cap Z) \subseteq V.$$

Then there exists $x \in K$ such that $x = Tx$.

This is a generalization of Tihonov's fixed point theorem since we can suppose in the latter case, that V is convex and so that $U = V$ in the above.

Let E be a linear space over the real or complex number field. The function $\| \, \|$: $E \to [0, \infty)$ will be said to be paranormed iff:

1. $\|x\|^* = 0 \Leftrightarrow x = 0$.
2. $\|-x\|^* = \|x\|^*$, for every $x \in E$.
3. $\|x + y\|^* \leqslant \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_0\|^* \to 0$ and $\lambda_n \to \lambda_0$, then $\|\lambda_n x_n - \lambda_0 x_0\|^* \to 0$.

The function ρ: $E \times E \to [0, \infty)$, defined by $\rho(x, y) = \|x - y\|^*$ ($x, y \in E$), is a distance function on E. If (E, ρ) is a complete metric space, it is a Fréchet space. Furthermore $(E, \| \, \|^*)$ is a topological vector space and its family of neighborhoods of zero is given by $\{V_\varepsilon\}_{\varepsilon > 0}$ where $V_\varepsilon = \{x \mid x \in E, \|x\|^* < \varepsilon\}$.

DEFINITION 1. *Let $(E, \| \, \|^*)$ be a paranormed space and K be a nonempty subset of E. We say that the set K satisfies Zima's condition if there exists $C > 0$ such that $\|\lambda x\|^* \leqslant C\lambda \|x\|^*$, for every $0 \leqslant \lambda \leqslant 1$ and every $x \in K - K$.*

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Zima in [7] has given an example of a space E and of a set K which satisfies the above condition.

We now proceed to a generalization of Rzepecki's fixed point theorem. (We should point out, however, that our proof reduces to an application of Rzepecki's theorem.)

THEOREM 2. *Let X be a Hausdorff topological vector space, K be a nonempty, closed and convex subset of X, T: $X \to X$ be an affine continuous mapping, and S: $K \to X$ be a continuous mapping such that $\overline{S(K)}$ is compact. Suppose that the following conditions are satisfied*:

(i) *For every $y \in \overline{\text{co}}\, S(K)$ there exists one and only one solution $x(y) \in K$ of the equation $z = Tz + y$ and the set $\{x(y)\}_{y \in \overline{S(K)}}$ is compact.*

(ii) *For every $V \in \mathfrak{U}$ and every $x \in \overline{S(K)}$ there exists $U \in \mathfrak{U}$ such that $\text{co}((x + U) \cap \overline{S(K)}) \subseteq x + V$, where $\mathfrak{U}$ is the base of the neighborhoods of zero in X.*

Then there exists $x \in K$ such that $x = Tx + Sx$.

PROOF. We first prove that the mapping R: $y \to x(y)$ $(y \in \overline{\text{co}}\, S(K))$ is continuous on the set $\overline{S(K)}$. Suppose that $\{y_\alpha\}_{\alpha \in \mathcal{Q}}$ is a net from $\overline{S(K)}$ such that $\lim_{\alpha \in \mathcal{Q}} y_\alpha = y$ and such that, for every $\alpha \in \mathcal{Q}$, $Ry_\alpha = TRy_\alpha + y_\alpha$. Since the set $\{x(y)\}_{y \in \overline{S(K)}}$ is compact, there exists a convergent subnet $\{Ry_{\alpha_\beta}\}$ of the net $\{Ry_\alpha\}$. Thus

$$\lim_\beta Ry_{\alpha_\beta} = T\Big(\lim_\beta Ry_{\alpha_\beta}\Big) + \lim_\beta y_{\alpha_\beta} = T\Big(\lim_\beta Ry_{\alpha_\beta}\Big) + y$$

and so $\lim_\beta Ry_{\alpha_\beta}$ is the solution of the equation $z = Tz + y$, which implies that $\lim_\beta Ry_{\alpha_\beta} = Ry$. Since each subnet of the net $\{Ry_\alpha\}$ has a convergent subnet with a limit Ry, it follows that $\lim_\alpha Ry_\alpha = Ry$. It is obvious that R^{-1} is continuous since

$$R^{-1}z = z - Tz \qquad \big(z \in R\big(\overline{\text{co}}\, S(K)\big)\big).$$

Next we prove that the mapping R is affine. Suppose that $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $x_1, x_2 \in \overline{\text{co}}\, \overline{S(K)}$. Then $Rx_1 = TRx_1 + x_1$, $Rx_2 = TRx_2 + x_2$ and so $\alpha Rx_1 + \beta Rx_2 = T(\alpha Rx_1 + \beta Rx_2) + \alpha x_1 + \beta x_2$ which implies that $R(\alpha x_1 + \beta x_2) = \alpha Rx_1 + \beta Rx_2$. Now, since R is affine, for every convex set $M \subseteq \overline{\text{co}}\, S(K)$ the set $R(M)$ is also convex. This implies that $R(\text{co}\, N)$ is convex and so $\text{co}\, R(\text{co}\, N) = R(\text{co}\, N)$. Since $R(N) \subseteq R(\text{co}\, N)$ it follows that $\text{co}\, R(N) \subseteq \text{co}\, R(\text{co}\, N) = R(\text{co}\, N)$. We define the mapping R^*: $K \to K$ in the following way:

$$R^*x = RSx, \quad \text{for every } x \in K.$$

We now show that the mapping R^* satisfies all the conditions of Rzepecki's fixed point theorem, where the set Z is taken to be the compact set $R(\overline{S(K)})$. Let $V \in \mathfrak{U}$ and $x \in R(\overline{S(K)})$. We shall prove that there exists $U \in \mathfrak{U}$ such that

$$\text{co}\big((x + U) \cap R\big(\overline{S(K)}\big)\big) \subseteq x + V, \quad \text{for every } x \in R\big(\overline{S(K)}\big).$$

Since $x \in R(\overline{S(K)})$, there exists $u \in \overline{S(K)}$ such that $x = Ru$. The mapping R is continuous at the point u and so there exists $V' \in \mathfrak{U}$ such that

$$R\big((u + V') \cap \overline{\text{co}}\, S(K)\big) \subseteq Ru + V.$$

Furthermore, from (ii) it follows that there exists $U' \in \mathfrak{U}$ such that

$$\text{(1)} \qquad \operatorname{co}\big((u + U') \cap \overline{S(K)}\big) \subseteq u + V'.$$

From (1) it follows that

$$R\big(\operatorname{co}\big((u + U') \cap \overline{S(K)}\big)\big) \subseteq R((u + V') \cap \overline{\operatorname{co}}\, S(K)) \subseteq Ru + V$$

and since R is a one-to-one mapping

$$\text{(2)} \qquad \operatorname{co}\big(R(u + U') \cap R\big(\overline{S(K)}\big)\big) \subseteq Ru + V.$$

The mapping R^{-1} is continuous, and so there exists $U \in \mathfrak{U}$ such that

$$R^{-1}(Ru + U) \subseteq R^{-1}(Ru) + U' = u + U',$$

and thus $(Ru + U) \cap R(\overline{S(K)}) \subseteq R((u + U') \cap (\overline{S(K)}))$. From (2) we conclude that

$$\operatorname{co}\big((Ru + U) \cap R\big(\overline{S(K)}\big)\big) \subseteq Ru + V4$$

and so the mapping R^* satisfies all the conditions of Theorem 1. This implies that $\operatorname{Fix}(R^*) \neq \varnothing$ and, since $\operatorname{Fix}(R^*) \subseteq \operatorname{Fix}(T + S)$, it follows that $\operatorname{Fix}(T + S) \neq \varnothing$.

COROLLARY 1. *Let $(E, \|\ \|^*)$ be a paranormed space and K be a nonempty, closed and convex subset of E. Let $T\colon E \to E$ be a continuous and affine mapping, $S\colon K \to E$ be a continuous mapping such that $\overline{S(K)}$ is compact and satisfies Zima's conditions, and suppose for every $y \in \overline{\operatorname{co}}\, S(K)$ there exists one and only one solution $x(y) \in K$ of the equation $z = Tz + y$ with $\{x(y)\}_{y \in \overline{S(K)}}$ compact. Then there exists $x \in K$ such that $x = Tx + Sx$.*

PROOF. It is easy to see that, since $\overline{S(K)}$ satisfies Zima's condition, the condition (ii) of Theorem 2 is satisfied and so there exists $x \in E$ such that $x = Tx + Sx$.

DEFINITION 2. [5] *Let (X, d) be a metric space and $T\colon X \to X$. The mapping $T\colon X \to X$ is a generalized contraction iff $d(Tx, Ty) \leqslant L(r, s)d(x, y)$, for every $x, y \in X$, $r \leqslant d(x, y) \leqslant s$, where the function L is defined for every $(r, s) \in (0, \infty) \times (0, \infty)$ such that $r \leqslant s$ and $L(r, s) < 1$.*

REMARK. The fixed point theorem of [5] for generalized contractions, which we use below, is also an immediate consequence of the fixed point theorem of A. Meir and E. Keeler [4].

From Corollary 1 we can derive the following corollary.

COROLLARY 2. *Let $(E, \|\ \|^*)$ be a paranormed space, K be a nonempty, convex and complete subset of E, $T\colon E \to E$ be an affine generalized contraction mapping, $S\colon K \to E$ be a compact mapping such that $T(K) + \overline{\operatorname{co}}\, S(K) \subseteq K$, the set $\overline{S(K)}$ satisfies Zima's condition and the set $(I - T)^{-1}\overline{S(K)}$ be bounded. Then there exists $x \in K$ such that $x = Tx + Sx$.*

PROOF. Since $T(K) + \overline{\operatorname{co}}\, S(K) \subseteq K$ and T is generalized contraction for each $y \in \overline{\operatorname{co}}\, S(K)$ there exists one and only one element $Ry \in K$ such that $Ry = TRy + y$ ([5]; cf. [4]). It remains to be proved that the set $\{Ry\}_{y \in \overline{S(K)}}$ is compact. To do this we shall show that the mapping R is continuous. Suppose that $\{x_n\}_{n \in N} \subseteq \overline{S(K)}$ and that $\lim_{n\to\infty} x_n = x$. If, on the contrary, the mapping R is not continuous, then there exists $\varepsilon > 0$ and a sequence $\{n(k)\}_{k\in N} \subseteq N$ such that

$$\| Rx_{n(k)} - Rx \|^* \geq \varepsilon \qquad (n(k) \geq k, \text{ for every } k \in N).$$

Since the set $\underline{(I - T)}^{-1}\overline{S(K)}$ is bounded, there exists $K' > 0$ such that $\| Ry \|^* \leq K'$, for every $y \in \overline{S(K)}$, and so for every $k \in N$,

$$\| Rx_{n(k)} - Rx \|^* \leq 2K'.$$

This in turn implies that

$$(3) \quad \| Rx_{n(k)} - Rx \|^* \leq L(\varepsilon, 2K')\| Rx_{n(k)} - Rx \|^* + \| x_{n(k)} - x \|^*, \qquad k \in N.$$

Since $\{\| Rx_{n(k)} - Rx \|^* \mid k \in N\} \subseteq [\varepsilon, 2K']$, there exists a subsequence $\{x_{n(k(r))}\}_{r\in N}$ such that

$$m = \lim_{r\to\infty} \| Rx_{n(k(r))} - Rx \|^*$$

and so, from (3), we have

$$m \leq L(\varepsilon, 2K')m < m$$

which is a contradiction.

We shall now give an application of Theorem 2 which refers to the existence of a solution to the equation $x = Tx + Sx$ in Φ-paranormed spaces **[2]**. We begin with some notations and definitions. We shall subsequently denote the set of all real numbers by R. Furthermore, let E be a vector space over $\mathcal{K}$ (real or complex number field) and R_Δ be the set of all mappings from Δ into R. The Tihonov product topology and the operations of $+$ and scalar multiplication are as usual. If $f, g \in R$ we say that $f \leq g$ iff $f(t) \leq g(t)$, for every $t \in \Delta$, and by P_Δ we shall denote the cone of nonnegative elements in R_Δ.

In **[2]** S. Kasahara introduced the following notion of paranormed spaces, which we shall call a Φ paranormed space.

DEFINITION 3. *The triplet* $(E, \| \ \|, \Phi)$ *is a* Φ *paranormed space iff* $\| \ \|: E \to P_\Delta$ *and* Φ *is a linear, continuous, positive mapping from* R_Δ *into* R_Δ *such that the following conditions are satisfied*:

1. $\| x \| = 0 \Leftrightarrow x = 0$.
2. $\| \lambda x \| = | \lambda | \, \| x \|$, *for every* $x \in E$ *and every* $\lambda \in \mathcal{K}$.
3. $\| x + y \| \leq \Phi(\| x \|) + \Phi(\| y \|)$, *for every* $x, y \in E$.

Let $\mathcal{U}$ denote the family of neighborhoods of zero in R_Δ. For each $U \in \mathcal{U}$ we denote the set $\{x \mid x \in E, \| x \| \in U\}$ by V_U. Then E is a topological vector space in which $\{V_U\}_{U\in\mathcal{U}}$ is the family of neighborhoods of zero in E.

In **[2]** it is proved that every Hausdorff topological vector space is a Φ paranormed space $(E, \| \ \|, \Phi)$ over a topological semifield R_Δ.

DEFINITION 4. *Let* $(E, \| \ \|, \Phi)$ *be a* Φ *paranormed space over a topological semifield* R_Δ *and* $K \subseteq E$. *If for every* $n \in N$, *every* $u_i \in K - K$ $(i = 1, 2, \ldots, n)$ *and* $(s_1, s_2, \ldots, s_n) \in R^n$ *such that* $s_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ *and* $\sum_{i=1}^n s_i = 1$,

$$\left\| \sum_{i=1}^{n} s_i u_i \right\| \leq \sum_{i=1}^{n} s_i \Phi(\| u_i \|),$$

we say that the set K is of Φ-type.

In [3] Matusov used Kasahara's result in order to prove a fixed point theorem.

Let $\mathcal{U}$ be the family of neighborhoods of zero in R_Δ and $U \in \mathcal{U}$. Then $\{x \mid x \in E, \|x\| \in U\}$ is a neighborhood of zero V_U in E and let us denote the family $\{V_U\}_{U \in \mathcal{U}}$ by $\mathcal{U}'$. Suppose now that K is a subset of E and that K is of Φ-type. We prove that for every $V \in \mathcal{U}'$ there exists $V' \in \mathcal{U}'$ such that for every $x \in K$

$$\operatorname{co}((x + V') \cap K) \subseteq x + V.$$

Since $V \in \mathcal{U}'$, there exists $\mu = \{t_1, t_2, \ldots, t_n\} \subseteq \Delta$ and $\varepsilon > 0$ such that

$$\|u\| \in U_{\mu,\varepsilon} \Rightarrow u \in V$$

where $U_{\mu,\varepsilon} = \{x \mid \|x\|(t) < \varepsilon$, for every $t \in \Delta\}$. Since the mapping Φ is linear and continuous there exists $V' = V_U$, such that

$$u \in V' \Rightarrow \Phi(\|u\|) \in U_{\mu,\varepsilon}.$$

It is easy to see that

$$\operatorname{co}((x + V') \cap K) \subseteq x + V, \quad \text{for every } x \in K.$$

Indeed, suppose that $u \in \operatorname{co}((x + V') \cap K)$. Then $u = \sum_{i=1}^n \lambda_i x_i$ where $x_i \in (x + V') \cap K$ $(i = 1,2,\ldots,n)$, $\lambda_i \geqslant 0$ $(i = 1,2,\ldots,n)$ and $\sum_{i=1}^n \lambda_i = 1$. Thus

$$\|u - x\|(t) = \left\| \sum_{i=1}^n \lambda(x_i - x) \right\|(t) \leqslant \sum_{i=1}^n \lambda_i \Phi(\|x_i - x\|)(t) \leqslant \varepsilon,$$

for every $t \in \mu$, and so $\|u - x\| \in U_{\mu,\varepsilon}$. This implies that $u - x \in V$ and so $u \in x + V$.

Now, we can formulate the following corollary.

COROLLARY 3. *Let $(X, \|\ \|, \Phi)$ be a Φ-paranormed space, K be a nonempty, closed and convex subset of the space X, T: $X \to X$ be an affine continuous mapping, and S: $K \to X$ be a continuous mapping such that $\overline{S(K)}$ is compact and of Φ-type. Suppose also that the condition* (i) *of Theorem* 2 *is satisfied. Then there exists $x \in K$ such that $x = Tx + Sx$.*

REFERENCES

1. O. Hadžić, *Fixed point theorems for multivalued mappings in topological vector spaces*, Glas. Mat. Ser. III **14** (33) (1980), 113–120.

2. S. Kasahara, *On formulations of topological linear spaces by topological semifields*, Math. Japon. **19** (1974), 121–134.

3. V. E. Matusov, *Obobshcheniye teoremii o nepodvizhnoi tochke Tikhonova*, Dokl. Akad. Nauk UzSSR **1978**, no. 2, 12–14.

4. A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

5. T. Riedrich, *Vorlesungen über nichtlineare Operatoren-gleichungen*, Teubner, Leipzig, 1976.

6. B. Rzepecki, *Remarks on Schauder's Fixed Point Theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **27** (1976), 473–480.

7. K. Zima, *On the Schauder's fixed point theorem with respect to paranormed space*, Comment. Math. Prace Mat. **19** (1977), 421–423.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOVI SAD, FACULTY OF SCIENCES, 21000 NOVI SAD, ULICA DR ILIJE DJURIČIĆA 4, YUGOSLAVIA