A FIXED POINT THEOREM
FOR THE SUM OF TWO MAPPINGS

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Abstract. A generalization of a fixed point theorem of Rzepecki is proved and it is shown that in a paranormed space $E$ this result yields, under certain circumstances, solutions to the equation $x = Tx + Sx$ for $T: E \to E$ either continuous and affine or a generalized contraction, and $S: K \subseteq E \to E$ compact.

In [7] Zima proved a generalization of the Schauder fixed point theorem in a paranormed space setting. (Paranormed spaces are nonlocally convex topological vector spaces; see the definition below.) B. Rzepecki then proved the following generalization of Zima’s result.

**Theorem 1 [6].** Let $X$ be a Hausdorff topological vector space, $K$ be a nonempty, closed and convex subset of $X$ and $T$ be a continuous mapping from $K$ into a compact set $Z$ ($Z \subseteq K$). Suppose that for every $x \in Z$ and every neighborhood $V$ of $x$ there exists a neighborhood $U$ of $x$ such that

$$\co(U \cap Z) \subseteq V.$$ 

Then there exists $x \in K$ such that $x = Tx$.

This is a generalization of Tihonov’s fixed point theorem since we can suppose in the latter case, that $V$ is convex and so that $U = V$ in the above.

Let $E$ be a linear space over the real or complex number field. The function $\| \|$ of $E \to [0, \infty)$ will be said to be paranormed iff:

1. $\| x \|^* = 0 \iff x = 0$.
2. $\| -x \|^* = \| x \|^*$, for every $x \in E$.
3. $\| x + y \|^* \leq \| x \|^* + \| y \|^*$, for every $x, y \in E$.
4. If $\| x_n - x_0 \|^* \to 0$ and $\lambda_n \to \lambda_0$, then $\| \lambda_n x_n - \lambda_0 x_0 \|^* \to 0$.

The function $\rho: E \times E \to [0, \infty)$, defined by $\rho(x, y) = \| x - y \|^*$ $(x, y \in E)$, is a distance function on $E$. If $(E, \rho)$ is a complete metric space, it is a Fréchet space. Furthermore $(E, \| \|)$ is a topological vector space and its family of neighborhoods of zero is given by $\{ V_{\varepsilon} \}_{\varepsilon > 0}$ where $V_\varepsilon = \{ x \mid x \in E, \| x \|^* < \varepsilon \}$.

**Definition 1.** Let $(E, \| \|)$ be a paranormed space and $K$ be a nonempty subset of $E$. We say that the set $K$ satisfies Zima’s condition if there exists $C > 0$ such that $\| \lambda x \|^* \leq C \| x \|^*$, for every $0 \leq \lambda \leq 1$ and every $x \in K - K$. 

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Zima in [7] has given an example of a space $E$ and of a set $K$ which satisfies the above condition.

We now proceed to a generalization of Rzepecki's fixed point theorem. (We should point out, however, that our proof reduces to an application of Rzepecki's theorem.)

**THEOREM 2.** Let $X$ be a Hausdorff topological vector space, $K$ be a nonempty, closed and convex subset of $X$, $T: X \to X$ be an affine continuous mapping, and $S: K \to X$ be a continuous mapping such that $S(K)$ is compact. Suppose that the following conditions are satisfied:

(i) For every $y \in \overline{co \ S(K)}$ there exists one and only one solution $x(y) \in K$ of the equation $z = Tz + y$ and the set $(x(y))_{y \in \overline{S(K)}}$ is compact.

(ii) For every $V \in \mathcal{U}$ and every $x \in S(K)$ there exists $U \in \mathcal{U}$ such that $co((x + U) \cap S(K)) \subseteq x + V$, where $\mathcal{U}$ is the base of the neighborhoods of zero in $X$.

Then there exists $x \in K$ such that $x = Tx + Sx$.

**Proof.** We first prove that the mapping $R: y \to x(y)$ ($y \in \overline{co \ S(K)}$) is continuous on the set $S(K)$. Suppose that $(y_\alpha)_{\alpha \in \mathcal{A}}$ is a net from $S(K)$ such that $lim_{\alpha \in \mathcal{A}} y_\alpha = y$ and such that, for every $\alpha \in \mathcal{A}$, $Ry_\alpha = TRy_\alpha + y_\alpha$. Since the set $(x(y))_{y \in \overline{S(K)}}$ is compact, there exists a convergent subnet $(Ry_\alpha^*)$ of the net $(Ry_\alpha)$. Thus

$$lim_{\beta} Ry_\alpha^* = T\left(lim_{\beta} Ry_\alpha^*\right) + lim_{\beta} y_\alpha^* = T\left(lim_{\beta} Ry_\alpha^*\right) + y$$

and so $lim_{\beta} Ry_\alpha^*$ is the solution of the equation $z = Tz + y$, which implies that $lim_{\beta} Ry_\alpha^* = Ry$. Since each subnet of the net $(Ry_\alpha)$ has a convergent subnet with a limit $Ry$, it follows that $lim_{\alpha} Ry_\alpha = Ry$. It is obvious that $R^{-1}$ is continuous since

$$R^{-1}z = z - Tz \quad (z \in R(\overline{co \ S(K)})).$$

Next we prove that the mapping $R$ is affine. Suppose that $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $x_1, x_2 \in \overline{co \ S(K)}$. Then $Rx_1 = TRx_1 + x_1$, $Rx_2 = TRx_2 + x_2$ and so $\alpha Rx_1 + \beta Rx_2 = T(\alpha Rx_1 + \beta Rx_2) + \alpha x_1 + \beta x_2$ which implies that $R(\alpha x_1 + \beta x_2) = \alpha Rx_1 + \beta Rx_2$. Now, since $R$ is affine, for every convex set $M \subseteq \overline{co \ S(K)}$ the set $R(M)$ is also convex. This implies that $R(\overline{co \ N})$ is convex and so $co R(\overline{co \ N}) = R(\overline{co \ N})$. Since $R(N) \subseteq R(\overline{co \ N})$ it follows that $co R(N) \subseteq co R(\overline{co \ N}) = R(\overline{co \ N})$. We define the mapping $R^*: K \to K$ in the following way:

$$R^*x = RSx, \quad \text{for every } x \in K.$$
Furthermore, from (ii) it follows that there exists \( U' \in \mathcal{K} \) such that

\[
\text{co}( (u + U') \cap S(K) ) \subseteq u + V'.
\]

From (1) it follows that

\[
R(\text{co}( (u + U') \cap S(K) )) \subseteq R((u + V') \cap \text{co} S(K)) \subseteq Ru + V
\]

and since \( R \) is a one-to-one mapping

\[
\text{co}(R(u + U') \cap R(S(K))) \subseteq Ru + V.
\]

The mapping \( R^{-1} \) is continuous, and so there exists \( U \in \mathcal{K} \) such that

\[
R^{-1}(Ru + U) \subseteq R^{-1}(Ru) + U = u + U',
\]

and thus \( (Ru + U) \cap R(S(K)) \subseteq R((u + U') \cap (S(K))). \) From (2) we conclude that

\[
\text{co}(((Ru + U) \cap R(S(K)))) \subseteq Ru + V'
\]

and so the mapping \( R^* \) satisfies all the conditions of Theorem 1. This implies that \( \text{Fix}(R^*) \neq \emptyset \) and, since \( \text{Fix}(R^*) \subseteq \text{Fix}(T + S) \), it follows that \( \text{Fix}(T + S) \neq \emptyset \).

**Corollary 1.** Let \((E, \| \|^*)\) be a paranormed space and \( K \) be a nonempty, closed and convex subset of \( E \). Let \( T: E \to E \) be a continuous and affine mapping, \( S: K \to E \) be a continuous mapping such that \( S(K) \) is compact and satisfies Zima’s conditions, and suppose for every \( y \in \overline{S(K)} \) there exists one and only one solution \( x(y) \in K \) of the equation \( z = Tx + y \) with \( \{x(y)\}_{y \in \overline{S(K)}} \) compact. Then there exists \( x \in K \) such that \( x = Tx + Sx \).

**Proof.** It is easy to see that, since \( \overline{S(K)} \) satisfies Zima’s condition, the condition (ii) of Theorem 2 is satisfied and so there exists \( U \in \mathcal{K} \) such that \( x = Tx + Sx \).

**Definition 2.** [5] Let \((X, d)\) be a metric space and \( T: X \to X \). The mapping \( T: X \to X \) is a generalized contraction iff \( d(Tx, Ty) \leq L(r, s)d(x, y) \), for every \( x, y \in X \), \( r \leq d(x, y) \leq s \), where the function \( L \) is defined for every \((r, s) \in (0, \infty) \times (0, \infty) \) such that \( r \leq s \) and \( L(r, s) < 1 \).

**Remark.** The fixed point theorem of [5] for generalized contractions, which we use below, is also an immediate consequence of the fixed point theorem of A. Meir and E. Keeler [4].

From Corollary 1 we can derive the following corollary.

**Corollary 2.** Let \((E, \| \|^*)\) be a paranormed space, \( K \) be a nonempty, convex and complete subset of \( E \), \( T: E \to E \) be an affine generalized contraction mapping, \( S: K \to E \) be a compact mapping such that \( T(K) + \text{co} S(K) \subseteq K \), the set \( \overline{S(K)} \) satisfies Zima’s condition and the set \((I - T)^{-1}\overline{S(K)}\) be bounded. Then there exists \( x \in K \) such that \( x = Tx + Sx \).

**Proof.** Since \( T(K) + \text{co} S(K) \subseteq K \) and \( T \) is generalized contraction for each \( y \in \text{co} S(K) \) there exists one and only one element \( Ry \in K \) such that \( Ry = TRy + y \) ([5]; cf. [4]). It remains to be proved that the set \( \{Ry\}_{y \in \text{co} S(K)} \) is compact. To do this we shall show that the mapping \( R \) is continuous. Suppose that \( \{x_n\}_{n \in \mathbb{N}} \subseteq S(K) \) and
that \( \lim_{n \to \infty} x_n = x \). If, on the contrary, the mapping \( R \) is not continuous, then there exists \( \varepsilon > 0 \) and a sequence \( \{n(k)\}_{k \in \mathbb{N}} \subseteq \mathbb{N} \) such that
\[
\| R x_{n(k)} - R x \|_* \geq \varepsilon \quad (n(k) \equiv k, \text{ for every } k \in \mathbb{N}).
\]
Since the set \( (I - T)^{-1} S(K) \) is bounded, there exists \( K' > 0 \) such that \( \| R y \|_* \leq K' \), for every \( y \in S(K) \), and so for every \( k \in \mathbb{N} \),
\[
\| R x_{n(k)} - R x \|_* \leq 2K'.
\]
This in turn implies that
\[
(3) \quad \| R x_{n(k)} - R x \|_* \leq L(\varepsilon, 2K') \| R x_{n(k)} - R x \|_* + \| x_{n(k)} - x \|_*, \quad k \in \mathbb{N}.
\]
Since \( \{\| R x_{n(k)} - R x \|_* | k \in \mathbb{N}\} \subseteq [\varepsilon, 2K'] \), there exists a subsequence \( \{x_{n(k(r))}\}_{r \in \mathbb{N}} \) such that
\[
m = \lim_{r \to \infty} \| R x_{n(k(r))} - R x \|_*
\]
and so, from (3), we have
\[
m \geq L(\varepsilon, 2K')m < m
\]
which is a contradiction.

We shall now give an application of Theorem 2 which refers to the existence of a solution to the equation \( x = T x + S x \) in \( \Phi \)-paranormed spaces [2]. We begin with some notations and definitions. We shall subsequently denote the set of all real numbers by \( R \). Furthermore, let \( E \) be a vector space over \( \mathbb{K} \) (real or complex number field) and \( R^{\Delta} \) be the set of all mappings from \( \Delta \) into \( R \). The Tihonov product topology and the operations of + and scalar multiplication are as usual. If \( f, g \in R \) we say that \( f \leq g \) iff \( f(t) \leq g(t) \), for every \( t \in \Delta \), and by \( P_{\Delta} \) we shall denote the cone of nonnegative elements in \( R^{\Delta} \).

In [2] S. Kasahara introduced the following notion of paranormed spaces, which we shall call a \( \Phi \) paranormed space.

**Definition 3.** The triplet \( (E, \| \|, \Phi) \) is a \( \Phi \) paranormed space iff \( \| \| : E \to P_{\Delta} \) and \( \Phi \) is a linear, continuous, positive mapping from \( R^{\Delta} \) into \( R^{\Delta} \) such that the following conditions are satisfied:

1. \( \| x \| = 0 \iff x = 0 \).
2. \( \| \lambda x \| = |\lambda| \| x \|, \text{ for every } x \in E \text{ and every } \lambda \in \mathbb{K} \).
3. \( \| x + y \| \leq \Phi(\| x \|) + \Phi(\| y \|), \text{ for every } x, y \in E \).

Let \( \mathcal{U} \) denote the family of neighborhoods of zero in \( R^{\Delta} \). For each \( U \in \mathcal{U} \) we denote the set \( \{x \in E | \| x \| \in U\} \) by \( V_U \). Then \( E \) is a topological vector space in which \( \{V_U\}_{U \in \mathcal{U}} \) is the family of neighborhoods of zero in \( E \).

In [2] it is proved that every Hausdorff topological vector space is a \( \Phi \) paranormed space \( (E, \| \|, \Phi) \) over a topological semifield \( R^{\Delta} \).

**Definition 4.** Let \( (E, \| \|, \Phi) \) be a \( \Phi \) paranormed space over a topological semifield \( R^{\Delta} \) and \( K \subseteq E \). If for every \( n \in \mathbb{N} \), every \( u_i \in K - K \) \( (i = 1, 2, \ldots, n) \) and \( (s_1, s_2, \ldots, s_n) \in R^n \) such that \( s_i \in [0, 1] \) \( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^n s_i = 1 \),
\[
\left\| \sum_{i=1}^n s_i u_i \right\| \leq \sum_{i=1}^n s_i \Phi(\| u_i \|),
\]
we say that the set \( K \) is of \( \Phi \)-type.
In [3] Matusov used Kasahara’s result in order to prove a fixed point theorem.

Let $\mathcal{U}$ be the family of neighborhoods of zero in $R^\Delta$ and $U \in \mathcal{U}$. Then \( \{x \mid x \in E, \|x\| \in U\} \) is a neighborhood of zero \( V_U \) in \( E \) and let us denote the family \( \{V_U\}_{U \in \mathcal{U}} \) by $\mathcal{U}'$. Suppose now that \( K \) is a subset of \( E \) and that \( K \) is of $\Phi$-type. We prove that for every \( V \in \mathcal{U}' \) there exists \( V' \in \mathcal{U}' \) such that for every \( x \in K \)

\[ \text{co}(\{x + V' \cap K\}) \subseteq x + V. \]

Since \( V \in \mathcal{U}' \), there exists \( \mu = \{t_1, t_2, \ldots, t_n\} \subseteq \Delta \) and \( \varepsilon > 0 \) such that

\[ \|u\| \in U_{\mu, \varepsilon} \Rightarrow u \in V \]

where \( U_{\mu, \varepsilon} = \{x \mid \|x\|(t) < \varepsilon, \text{ for every } t \in \Delta\} \). Since the mapping $\Phi$ is linear and continuous there exists \( V'' = V_U \), such that

\[ u \in V'' = \Phi(\|u\|) \in U_{\mu, \varepsilon}. \]

It is easy to see that

\[ \text{co}(\{(x + V') \cap K\}) \subseteq x + V, \text{ for every } x \in K. \]

Indeed, suppose that \( u \in \text{co}(\{(x + V') \cap K\}) \). Then \( u = \sum_{i=1}^{n} \lambda_i x_i \) where \( x_i \in (x + V') \cap K (i = 1, 2, \ldots, n) \), \( \lambda_i > 0 \) (\( i = 1, 2, \ldots, n \)) and \( \sum_{i=1}^{n} \lambda_i = 1 \). Thus

\[ \|u - x\|(t) = \left\| \sum_{i=1}^{n} \lambda_i (x_i - x) \right\|(t) \leq \sum_{i=1}^{n} \lambda_i \Phi(\|x_i - x\|)(t) \leq \varepsilon, \]

for every \( t \in \mu \), and so \( \|u - x\| \in U_{\mu, \varepsilon} \). This implies that \( u - x \in V \) and so \( u \in x + V \).

Now, we can formulate the following corollary.

**Corollary 3.** Let \( (X, \|\|, \Phi) \) be a $\Phi$-paranormed space, \( K \) be a nonempty, closed and convex subset of the space \( X \), \( T \colon X \to X \) be an affine continuous mapping, and \( S \colon K \to X \) be a continuous mapping such that \( S(K) \) is compact and of $\Phi$-type. Suppose also that the condition (i) of Theorem 2 is satisfied. Then there exists \( x \in K \) such that \( x = Tx + Sx \).

**References**


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