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THE BIDUAL OF $C(X, E)$

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ABSTRACT. In this article we obtain, under certain conditions, a characterization of the bidual of the space $C(X, E)$ of continuous functions on a compact Hausdorff space X to a Banach space E. It is shown that if X is dispersed and E is arbitrary, or if X is arbitrary and E^* has the Radon-Nikodym property, then $C(X, E)^{**}$ can be represented as a space of continuous functions on a compact Hausdorff space Z to E^{**} when the latter space is given its weak* topology.

1. Introduction. Through this article, the letters E and F will denote Banach spaces, while X and Z stand for compact Hausdorff spaces. Given E, K will denote the one-dimensional Banach space consisting of the corresponding field of scalars. The interaction between elements of a Banach space E and those of its dual space is denoted by $\langle \cdot\, , \cdot \rangle$. We will write $E \simeq F$ to indicate that the Banach spaces E and F are isometric. $\mathfrak{L}(E; F)$ is the space of bounded linear operators on E to F.

Given X and E, $C(X, E)$ will denote the space of continuous functions on X to E provided with the supremum norm. If F is a dual space, $C(X, (F, \sigma^*))$ stands for the Banach space of continuous functions G on X to F when this latter space is provided with its weak* topology, again normed by $\|G\|_\infty = \sup_{x \in X}\|G(x)\|$. In the case of scalar-valued functions, a characterization of the bidual of $C(X, K)$ was first obtained by Kakutani [7], who showed that $C(X, K)^{**}$ is of the form $C(Z, K)$, for a certain compact Hausdorff space Z. This result has also been derived and discussed by other authors using various techniques—see, for example, [**1**, **5**, and **8**]. In [**1**, **5**, **7** and **8**] it is assumed that K is the real field $\mathbf{R}$, but proofs for $K = \mathbf{C}$ have been circulated among functional analysts for at least two decades.

The characterization of the first dual of $C(X, K)$ is of course given by the Riesz representation theorem, which states that $C(X, K)^*$ is the space $M(X, K)$ of regular Borel measures μ on X to K with finite variation $|\mu|$. The vector analogue of this result was provided by I. Singer, who showed that $C(X, E)^*$ is the space $M(X, E^*)$ of all regular Borel vector measures m on X to E^*, with finite variation $|m|$ [**10**]. An English version of this result can be found in [**11**, p. 192] or [**4**, p. 397]. In this paper we give, under certain restrictions on X and E, a characterization of $C(X, E)^{**}$ as a space of continuous vector-valued functions, providing a vector analogue of Kakutani's result.

Given any X and F, $M(X, F)$ will always denote the Banach space of regular Borel vector measures m on X to F with finite variation $|m|$. There is a natural

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injection of $M(X, K) \otimes F$ into $M(X, F)$ determined by $\mu \otimes \phi \to \mu(\cdot)\phi$ for $\mu \in M(X, K)$ and $\phi \in F$. In our notation we identify $M(X, K) \otimes F$ with its image under this injection, thus treating $M(X, K) \otimes F$ as a subspace of $M(X, F)$. In §3 we prove that under the condition

$$M(X, K) \otimes E^* \quad \text{is dense in} \quad M(X, E^*),$$

then $C(X, E)^{**}$ is of the form $C(Z, (E^{**}, \sigma^*))$ (Theorem 2). In §2 we obtain conditions on a compact space X and any Banach space F which insure that $[M(X, K) \otimes F]^- = M(X, F)$. It is shown this equality always holds if X is dispersed, or if F has the Radon-Nikodym property.

Given a measure space (Y, Σ, μ), we will denote the space $L^1(Y, \Sigma, \mu, K)$ by $L^1(\mu, K)$. And the space of (equivalence classes of) μ-integrable functions on Y to a Banach space F will be denoted by $L^1(\mu, F)$. For a measurable space (Y, Σ), $m(Y, \Sigma, K)$ denotes the Banach lattice of all measures μ: $\Sigma \to K$. For measures μ and ν, $\nu \ll \mu$ means that ν is absolutely continuous with respect to μ. For $\mu \in m(Y, \Sigma, K)$ and $A \in \Sigma$, $P_A\mu$ will denote that element of $m(Y, \Sigma, K)$ defined by $P_A\mu(B) = \mu(A \cap B)$, for $B \in \Sigma$. Given compact X, β will denote the σ-algebra of Borel subsets of X. We note for future reference that $M(X, K)$ is a closed ideal in $m(X, \beta, K)$ [**9**, p. 46].

If (Y, Σ, μ) is any measure space, we denote by $M(\mu)$ the closed ideal in $m(Y, \Sigma, \mathbf{R})$ consisting of all $\nu \in m(Y, \Sigma, \mathbf{R})$ such that $\nu \ll \mu$. And for a given Banach space F, we denote by $M(\mu, F)$ the space of all μ-continuous, F-valued measures on Σ with finite variation.

Throughout, scalar measures are denoted by μ, ν and vector measures by m. Facts about vector measures used in this paper are found in [**3**] and [**4**]. Our notation and terminology concerning lattice theory are consistent with that of [**9**]. In notation concerning tensor products and their completions we follow Chapter 8 of [**3**].

2. Topological and Banach space considerations.

LEMMA. *Given measurable spaces (Y, Σ) and (Y', Σ'), let M and M' be closed ideals in $m(Y, \Sigma, \mathbf{R})$ and $m(Y', \Sigma', \mathbf{R})$ respectively, and assume that there exists an isometry T mapping M onto M' such that T is order preserving. Then for each Banach space F (real or complex) there exists an isometry T_F mapping M_F onto M'_F, where M_F (resp. M'_F) denotes the Banach space of all F-valued measures m on Σ (resp. Σ'), with finite variation $|m|$, such that $|m| \in M$ (resp. M'). Moreover, T_F has the property that $|T_F m| = T|m|$ for each $m \in M_F$.*

PROOF. Let μ be a positive element of M and let $A' \in \Sigma'$ be given. We claim first of all that the measure $\mu_{A'}$ defined by $\mu_{A'} = T^{-1}(P_{A'}T\mu)$ is equal to $P_A\mu$ for some $A \in \Sigma$. For $T\mu = P_{A'}T\mu + P_{Y'-A'}T\mu$, and the elements on the right-hand side are disjoint. Since T^{-1}, like T, is order preserving, it preserves disjointness and thus $\mu = T^{-1}T\mu = \mu_{A'} + \mu_{Y'-A'}$, with the summands on the right again disjoint. We let A and $Y - A$ be a Hahn decomposition for the measure $\nu = \mu_{A'} - \mu_{Y'-A'}$ [**6**, p. 121]—i.e. A is positive and $Y - A$ is negative with respect to ν. Then $\mu_{A'} \leqslant \mu_{Y'-A'}$

on $Y - A$, and since trivially $\mu_{A'} \leqslant \mu_{A'}$ on $Y - A$, we have $P_{Y-A}\mu_{A'} \leqslant P_{Y-A}(\mu_{A'} \wedge \mu_{Y'-A'}) = 0$. Hence $\mu_{A'} = P_A\mu_{A'}$. Similarly $\mu_{Y'-A'} = P_{Y-A}\mu_{Y'-A'}$ so that the equation $\mu = \mu_{A'} + \mu_{Y'-A'}$ gives $\mu_{A'} = P_A\mu$.

We define $\Phi_\mu \colon \Sigma' \to \Sigma$ by $\Phi_\mu(A') = A$, where A' and A are related as above. It is clear that Φ_μ is well defined up to μ-equivalence, and it is a straightforward matter to check that Φ_μ is a Boolean algebra isomorphism of $\Sigma'/T\mu$ (Σ' modulo $T\mu$-null sets) onto Σ/μ.

Now in $m(Y, \Sigma, \mathbf{R})$, $\mu^{\perp\perp} = \{\nu \in m(Y, \Sigma, \mathbf{R}) \colon \nu \ll \mu\}$ [**9**, p. 45]. And the fact that M is an ideal insures that the inf in M of two elements belonging to M is also their inf in $m(Y, \Sigma, \mathbf{R})$. Thus the bipolar in M of a subset of M is the intersection of M with the bipolar taken in $m(Y, \Sigma, \mathbf{R})$. From this, and the fact that T preserves the relation of orthogonality, it follows that if ν is a positive element of M with $\nu \ll \mu$, then $T\nu \ll T\mu$. Thus if A and A' are as above, the fact that we have $P_{A'}T\nu \ll P_{A'}T\mu$ implies that

$$\nu_{A'} = T^{-1}(P_{A'}T\nu) \ll T^{-1}(P_{A'}T\mu) = P_A\mu.$$

From this it follows readily that $\Phi_\nu(A') \cap (Y - A)$ and $\Phi_\nu(Y' - A') \cap A$ are ν-null sets so that $\Phi_\nu(A') = \Phi_\mu(A')$ a.e. ν.

Next let $m \in M_F$. Then the equation $T_F m = m \circ \Phi_{|m|}$ defines a vector measure $T_F m$ on Σ'. And by what we have established in the preceding paragraph it follows that

$$T_F m = m \circ \Phi_\mu \tag{1}$$

for all positive $\mu \in M$ with $|m| \ll \mu$. Obviously we have $|m \circ \Phi_{|m|}| = |m| \circ \Phi_{|m|}$. Moreover $T|m| = |m| \circ \Phi_{|m|}$, since for positive measures $\mu \in M$ and $A' \in \Sigma'$ we have

$$\begin{aligned}\mu \circ \Phi_\mu(A') &= \mu\big(\Phi_\mu(A')\big) = P_{\Phi_\mu(A')}\mu(Y) = \| P_{\Phi_\mu(A')}\mu \| \\ &= \| P_{A'}T\mu \| = P_{A'}T\mu(Y') = T\mu(A').\end{aligned}$$

It thus follows that

$$|T_F m| = T|m|. \tag{2}$$

It remains to show that T_F is linear and maps M_F onto M'_F. Thus given $m_1, m_2 \in M_F$ we set $\mu = |m_1| + |m_2|$. Then by (1) we have

$$T_F(m_1 + m_2) = (m_1 + m_2) \circ \Phi_\mu = m_1 \circ \Phi_\mu + m_2 \circ \Phi_\mu = T_F m_1 + T_F m_2.$$

Clearly for $m \in M_F$ and scalars λ, we have $T_F \lambda m = \lambda T_F m$. Finally, T_F is surjective since it has $(T^{-1})_F$ as its inverse. For given $m' \in M'_F$ we have $T_F((T^{-1})_F m') = ((T^{-1})_F m') \circ \Phi_{|(T^{-1})_F m'|}$, which by (2) is equal to

$$\big((T^{-1})_F m'\big) \circ \Phi_{T^{-1}|m'|} = m' \circ \Phi'_{|m'|} \circ \Phi_{T^{-1}|m'|},$$

where $\Phi'_{|m'|}$ is the Boolean algebra isomorphism of $\Sigma/T^{-1}|m'|$ onto $\Sigma'/|m'|$ determined by the equation

$$T\big(P_A T^{-1}|m'|\big) = P_{\Phi'_{|m'|}(A)}|m'| \quad \text{for } A \in \Sigma/T^{-1}|m'|.$$

A check that $\Phi'_{|m'|}$ is equal to $\Phi^{-1}_{T^{-1}|m'|}$ then completes the proof of the lemma.

Corollary 1. *Given any compact Hausdorff space X and Banach space F, there exists a measure space (Y, Σ, μ) such that $M(X, F)$ is isometric to $M(\mu, F)$.*

Proof. $M(X, \mathbf{R})$ is an abstract L^1 space. Hence [**9**, p. 135], there exists a measure space (Y, Σ, μ) such that $M(X, \mathbf{R})$ is isometric and lattice isomorphic to the space $L^1(\mu, \mathbf{R})$. The measure space (Y, Σ, μ) is not, in general, σ-finite. However (see the proof of Theorem 2 on pp. 135–136 in [**9**]), the space is "strictly localizable" by which we mean that Y is the disjoint union of measurable sets Y_i, $i \in I$, with $\mu(Y_i) = 1$ for each i, and μ is the sum of its restrictions to the sets Y_i—$\mu(B) = \Sigma_{i \in I} \mu(Y_i \cap B)$ for all $B \in \Sigma$. Clearly the indefinite integral defines an isometric embedding of $L^1(\mu, \mathbf{R})$ into $M(\mu)$. And if $\nu \in M(\mu)$, since $|\nu|$ is finite there are at most countably many Y_i with $|\nu|(Y_i) \neq 0$. Thus the Radon-Nikodym theorem [**6**, p. 128] implies that $L^1(\mu, \mathbf{R})$ is isometric and lattice isomorphic to $M(\mu)$. We thus have established the existence of an isometry of $M(X, \mathbf{R})$ onto $M(\mu)$ which is order preserving. If, in the notation of the lemma, we take (Y, Σ) to be the (Y, Σ) of this corollary, (Y', Σ') to be $(X, \mathfrak{B})$, and take $M = M(\mu)$ and $M' = M(X, \mathbf{R})$, then since a vector measure m is regular iff its variation $|m|$ is regular, it follows that $M_F = M(\mu, F)$ and $M'_F = M(X, F)$, thus completing the proof.

Corollary 2. *Let X be compact Hausdorff and F a Banach space. Then $M(X, K) \hat{\otimes} F$ can be embedded in $M(X, F)$ in such a way that $\nu \otimes \phi$ corresponds to $\nu(\cdot)\phi$ for all $\nu \in M(X, K)$ and $\phi \in F$.*

Proof. Since $M(X, K) \simeq L^1(\mu, K)$, for the measure μ of [**9**, p. 135], $M(X, K) \hat{\otimes} F$ is isometric to $L^1(\mu, K) \hat{\otimes} F$ which is isometric to $L^1(\mu, F)$ [**3**, p. 228]. $L^1(\mu, F)$ is canonically embedded in $M(\mu, F)$ which is isometric to $M(X, F)$ by Corollary 1. One readily checks that $\nu \otimes \phi$ is mapped onto $\nu(\cdot)\phi$ by this sequence of isometries.

Theorem 1. *Let X be a compact Hausdorff space.*

(a) *If X is dispersed, then $M(X, F) = [M(X, K) \otimes F]^-$ for all Banach spaces F.*

(b) *If X is not dispersed, then $M(X, F) = [M(X, K) \otimes F]^-$ iff F has the Radon-Nikodym property.*

Proof. (a) follows directly from [**9**, p. 52].

For (b), first assume that F has the Radon-Nikodym property. Then the canonical embedding J: $L^1(\mu, F) \to M(\mu, F)$ is onto. For although the Radon-Nikodym property is usually formulated with respect to finite measure spaces [**3**, p. 61], the fact that our measure space is strictly localizable gives the desired result via an argument analogous to that used in the proof of Corollary 1. Thus the embedding of Corollary 2 is onto. Since $M(X, K) \otimes F$ viewed as a subspace of $M(X, K) \hat{\otimes} F$ is dense in $M(X, K) \hat{\otimes} F$, $M(X, K) \otimes F$ as a subspace of $M(X, F)$ is dense in this latter space.

Conversely, suppose that $[M(X, K) \otimes F]^- = M(X, F)$. Then the embedding of Corollary 2 is onto and thus J is surjective. That is, F has the Radon-Nikodym property with respect to μ. Since X is not dispersed, by [**9**, p. 52] there is a purely nonatomic measure in $M(X, K)$, and thus in the isometries established in [**9**, pp. 135–136],

$$M(X, K) \simeq L^1(\mu, K) \simeq \left[l^1(\Gamma, K) \oplus \left(\oplus \sum_{\alpha \in I'} L^1(\nu^\alpha, K) \right)_1 \right]_1,$$

where I' is a set of infinite cardinal numbers α, and ν^α is Lebesgue product measure on $[0, 1]^\alpha$, the index set I' cannot be void. Hence there exists a measurable subset $Y_i \subseteq Y$ with $\chi_{Y_i} \cdot L^1(\mu, K) \simeq L^1(\nu^\alpha, K)$ for some α. Thus, if Σ_i is the σ-algebra given by $\Sigma_i = \{A \cap Y_i: A \in \Sigma\}$, $\mu|_{\Sigma_i}$ is purely nonatomic and F has the Radon-Nikodym property with respect to $\mu|_{\Sigma_i}$. Thus, by [**2**, p. 26], F has the Radon-Nikodym property with respect to Lebesgue measure on [0, 1]. Consequently [**3**, p. 138] F has the Radon-Nikodym property.

3. Characterization of the bidual. We now characterize the bidual of $C(X, E)$ under the assumption that $M(X, K) \otimes E^*$ is dense in $M(X, E^*)$, which we have just seen is equivalent to the assumption that X is dispersed or E^* has the Radon-Nikodym property.

THEOREM 2. *Let X be a compact Hausdorff space and E a Banach space with $M(X, E^*) = [M(X, K) \otimes E^*]^-$. Then $C(X, E)^{**}$ is isometric to $C(Z, (E^{**}, \sigma^*))$, where Z is that compact Hausdorff space such that $C(X, K)^{**} \simeq C(Z, K)$.*

PROOF. By Singer's result $C(X, E)^* \simeq M(X, E^*)$. And we know by Corollary 2 that $M(X, K) \hat{\otimes} E^*$ is isometric to the closure of $M(X, K) \otimes E^*$ in $M(X, E^*)$. Hence $M(X, K) \hat{\otimes} E^* \simeq M(X, E^*)$ by our hypothesis. Thus

$$C(X, E)^{**} \simeq M(X, E^*)^* \simeq \left[M(X, K) \hat{\otimes} E^* \right]^* \simeq \left[E^* \hat{\otimes} M(X, K) \right]^*.$$

But by [**3**, p. 230], this latter space is isometric to $\mathfrak{L}(E^*; M(X, K)^*) \simeq \mathfrak{L}(E^*; C(Z, K))$.

We define a map T from $\mathfrak{L}(E^*; C(Z, K))$ to the space of functions on Z to E^{**} by

$$\langle \phi, (T\Psi)(z) \rangle = (\delta_z \circ \Psi)(\phi), \quad \text{for } \phi \in E^*, z \in Z,$$

and $\Psi \in \mathfrak{L}(E^*; C(Z, K))$, where δ_z denotes the point evaluation at $z \in Z$. Clearly, for fixed $z \in Z$, $(T\Psi)(z)$ is a linear functional on E^* with, for $\phi \in E^*$,

$$|\langle \phi, (T\Psi)(z) \rangle| = |(\Psi\phi)(z)| \leq \|\Psi\phi\|_\infty \leq \|\Psi\| \, \|\phi\|.$$

That is, $(T\Psi)(z) \in E^{**}$ and $\|(T\Psi)(z)\| \leq \|\Psi\|$. Clearly as a function from Z to E^{**} $(T\Psi)(\cdot)$ is weak* continuous, since for all $\phi \in E^*$, $\langle \phi, (T\Psi)(\cdot) \rangle = (\Psi\phi)(\cdot) \in C(Z, K)$. Hence T is a mapping from $\mathfrak{L}(E^*; C(Z, K))$ to $C(Z, (E^{**}, \sigma^*))$. Obviously T is linear and we have already seen that

$$\|T\Psi\|_\infty = \sup_{z \in Z} \|(T\Psi)(z)\| \leq \|\Psi\|, \text{ i.e., } \|T\| \leq 1.$$

We wish to show that T is an isometry of $\mathfrak{L}(E^*; C(Z, K))$ onto $C(Z, (E^{**}, \sigma^*))$. To this end we first show that T is norm-preserving. Suppose we are given $\varepsilon > 0$ and

$\Psi \in \mathcal{L}(E^*; C(Z, K))$, $\Psi \neq 0$. Choose $\phi \in E^*$ with $\|\phi\| = 1$ such that $\|\Psi\phi\|_\infty \geqslant \|\Psi\| - \varepsilon$. Then choose $z \in Z$ such that $|(\Psi\phi)(z)| = \|\Psi\phi\|_\infty$. Then

$$\|T\Psi\|_\infty \geqslant \|(T\Psi)(z)\| \geqslant |\langle\phi, (T\Psi)(z)\rangle| = |(\Psi\phi)(z)| \geqslant \|\Psi\| - \varepsilon.$$

Thus T is isometric.

Finally, we show that T is surjective. Let $G \in C(Z, (E^{**}, \sigma^*))$ be given. Define Ψ on E^* by $(\Psi\phi)(z) = \langle\phi, G(z)\rangle$ for $\phi \in E^*$ and $z \in Z$. Clearly $(\Psi\phi)(\cdot) \in C(Z, K)$ and Ψ is a linear map from E^* into $C(Z, K)$. It is bounded since

$$\|\Psi\phi\|_\infty = \sup_{z \in Z} |\langle\phi, G(z)\rangle| \leqslant \|G\|_\infty \|\phi\|.$$

Finally, using first the definition of T, then that of Ψ, we have, for all $z \in Z$ and $\phi \in E^*$, $\langle\phi, (T\Psi)(z)\rangle = (\Psi\phi)(z) = \langle\phi, G(z)\rangle$. That is, $T\Psi = G$.

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