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AN ELEMENTARY PROOF ABOUT THE ORDER OF THE ELEMENTS IN A DISCRETE GROUP

G. CROMBEZ

ABSTRACT. We give an elementary direct proof of the following property: if for a discrete group G some $l_p(G)$-space $(1 < p < \infty)$ is an algebra, then all elements of G have uniformly bounded order.

If G is a discrete group and $l_p(G)$ $(1 < p < \infty)$ is an algebra under convolution, then the property that all elements of G have uniformly bounded order is usually proved in an indirect way, by first showing that G is a Burnside group **[1]** (i.e., for any Haar measure μ on G there exists a constant $C_\mu > 0$ such that $\mu(AB) \geqslant C_\mu \mu(A)\mu(B)$ for all compact subsets A, B of G), and then using the special properties of a Burnside group. We give here an elementary direct proof of the mentioned property. We make use of the well-known fact that, if for a locally compact group G and some p $(1 < p < \infty)$ $L_p(G)$ is an algebra under convolution, there exists a constant $C > 0$ such that $\|f * g\|_p \leqslant C\|f\|_p\|g\|_p$ $(f, g \in L_p(G))$.

LEMMA. *For* $1 < p < \infty$ *we have*

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + (n-1)^p + n^p + (n-1)^p + \cdots + 1^p}{n^2} = \infty.$$

PROOF. The result is certainly true for $p = 2$ (since then the numerator is of order n^3), and so also for $p > 2$; on the contrary, it is not true for $p = 1$. In order to prove it for $1 < p < 2$ it is sufficient to prove the inequality

$$(1) \qquad 1^p + 2^p + \cdots + (n-1)^p + n^p + (n-1)^p + \cdots + 1^p > \frac{n^p(n-2)}{2},$$

for sufficiently great values of n $(1 < p < 2)$, and (1) will be true as soon as

$$(2) \qquad 1^p + 2^p + \cdots + (n-1)^p > \frac{n^p(n-4)}{4}, \quad \text{for } n \geqslant 5.$$

Now the truth of (2) may be shown by induction. For (2) is certainly true for $n = 5$. So, assuming that

$$1^p + 2^p + \cdots + (n-2)^p > \frac{(n-1)^p(n-1-4)}{4},$$

we have

$$1^p + 2^p + \cdots + (n-2)^p + (n-1)^p > \frac{(n-1)^p(n-1)}{4},$$

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and hence (2) will be true as soon as $(n-1)^p(n-1) > n^p(n-4)$, or

$$(1-1/n)^p > 1-3/(n-1). \tag{3}$$

But (3) is valid since $(1-1/n)^p > (1-1/n)^2$ for $1 < p < 2$, and since it is obvious that $(1-1/n)^2 > 1-3/(n-1)$. □

Let then G be a discrete group with identity e such that $l_p(G)$ is an algebra $(1 < p < \infty)$, and suppose that the elements of G do not have uniformly bounded order. Then, for each $n \in Z^+$ there exists a $y \in G$ such that $e \notin \{y, y^2, \ldots, y^{2n}\}$. Put $f = \sum_{i=1}^n \delta_{y^i}$, where $\delta_{y^i}(x) = 1$ for $x = y^i$ and zero in the other points of G. Then $\|f\|_p = n^{1/p}$, while a calculation shows that $(f*f)(y^2) = 1$, $(f*f)(y^3) = 2, \ldots, (f*f)(y^{n+1}) = n, \ldots, (f*f)(y^{2n}) = 1$, and $(f*f)(x) = 0$ when $x \notin \{y^2, y^3, \ldots, y^{2n}\}$. Hence $\|f*f\|_p = (1^p + \cdots + n^p + \cdots + 1^p)^{1/p}$, and so

$$\frac{\|f*f\|_p}{\|f\|_p\|f\|_p} = \left(\frac{1^p + 2^p + \cdots + n^p + (n-1)^p + \cdots + 1^p}{n^2}\right)^{1/p} \to \infty \quad \text{for } n \to \infty,$$

a contradiction.

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SEMINAR OF HIGHER ANALYSIS, STATE UNIVERSITY OF GHENT, GALGLAAN 2, B-9000 GHENT, BELGIUM