AN ELEMENTARY PROOF ABOUT THE ORDER OF THE ELEMENTS IN A DISCRETE GROUP

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Abstract. We give an elementary direct proof of the following property: if for a discrete group $G$ some $l_p(G)$-space $(1 < p < \infty)$ is an algebra, then all elements of $G$ have uniformly bounded order.

If $G$ is a discrete group and $l_p(G)$ $(1 < p < \infty)$ is an algebra under convolution, then the property that all elements of $G$ have uniformly bounded order is usually proved in an indirect way, by first showing that $G$ is a Burnside group [1] (i.e., for any Haar measure $\mu$ on $G$ there exists a constant $C_\mu > 0$ such that $\mu(AB) \geq C_\mu \mu(A)\mu(B)$ for all compact subsets $A$, $B$ of $G$), and then using the special properties of a Burnside group. We give here an elementary direct proof of the mentioned property. We make use of the well-known fact that, if for a locally compact group $G$ and some $p$ $(1 < p < \infty)$ $L_p(G)$ is an algebra under convolution, there exists a constant $C > 0$ such that $\|f * g\|_p \leq C \|f\|_p \|g\|_p$ ($f, g \in L_p(G)$).

**Lemma.** For $1 < p < \infty$ we have

\[
\lim_{n \to \infty} \frac{1^p + 2^p + \cdots + (n-1)^p + n^p + (n-1)^p + \cdots + 1^p}{n^2} = \infty.
\]

**Proof.** The result is certainly true for $p = 2$ (since then the numerator is of order $n^3$), and so also for $p > 2$; on the contrary, it is not true for $p = 1$. In order to prove it for $1 < p < 2$ it is sufficient to prove the inequality

(1) \[ 1^p + 2^p + \cdots + (n-1)^p + n^p + (n-1)^p + \cdots + 1^p > \frac{n^p(n-2)}{2}, \]

for sufficiently great values of $n$ $(1 < p < 2)$, and (1) will be true as soon as

(2) \[ 1^p + 2^p + \cdots + (n-1)^p > \frac{n^p(n-4)}{4}, \quad \text{for } n \geq 5. \]

Now the truth of (2) may be shown by induction. For (2) is certainly true for $n = 5$. So, assuming that

\[ 1^p + 2^p + \cdots + (n-2)^p > \frac{(n-1)^p(n-1)}{4}, \]

we have

\[ 1^p + 2^p + \cdots + (n-2)^p + (n-1)^p > \frac{(n-1)^p(n-1)}{4}. \]
and hence (2) will be true as soon as \((n - 1)^p(n - 1) > n^p(n - 4)\), or

\[(3) \quad (1 - 1/n)^p > 1 - 3/(n - 1).\]

But (3) is valid since \((1 - 1/n)^p > (1 - 1/n)^2\) for \(1 < p < 2\), and since it is obvious that \((1 - 1/n)^2 > 1 - 3/(n - 1)\). ∎

Let then \(G\) be a discrete group with identity \(e\) such that \(\ell_p(G)\) is an algebra \((1 < p < \infty)\), and suppose that the elements of \(G\) do not have uniformly bounded order. Then, for each \(n \in \mathbb{Z}^+\) there exists a \(y \in G\) such that \(e \notin \{y, y^2, \ldots, y^{2^n}\}\). Put \(f = \sum_{i=1}^{2^n} \delta_i, i\), where \(\delta_i, i(x) = 1\) for \(x = y^i\) and zero in the other points of \(G\). Then \(\|f\|_p = n^{1/p}\), while a calculation shows that \((f * f)(y^2) = 1\), \((f * f)(y^3) = 2\), \(\ldots\), \((f * f)(y^{n+1}) = n\), \(\ldots\), \((f * f)(y^{2n}) = 1\), and \((f * f)(x) = 0\) when \(x \notin \{y^2, y^3, \ldots, y^{2n}\}\). Hence \(\|f * f\|_p = (1^p + \cdots + n^p + \cdots + 1^p)^{1/p}\), and so

\[
\frac{\|f \ast f\|_p}{\|f\|_p \|f\|_p} = \left( \frac{1^p + 2^p + \cdots + n^p + (n - 1)^p + \cdots + 1^p}{n^2} \right)^{1/p} \to \infty \quad \text{for} \ n \to \infty,
\]

a contradiction.

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REFERENCES


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