PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

A NOTE ON HOLOMORPHIC IMBEDDINGS OF THE CLASSICAL CARTAN DOMAINS INTO THE UNIT BALL

YOSHIHISA KUBOTA

ABSTRACT. Let D be a classical Cartan domain and let B be the unit ball. We find the exact value of the supremum of the set of positive numbers ρ satisfying the condition: $\rho B \subset f(D)$ for a certain holomorphic imbedding $f: D \to B$.

1. This paper is concerned with the following problem: Let D be a bounded homogeneous domain in $\mathbf{C}^n$. By $\mathbb{S}(D)$ we denote the family of holomorphic imbeddings f of D into the unit ball B_n in $\mathbf{C}^n$ such that $f(D) \ni 0$. For each $f \in \mathbb{S}(D)$ we define

$$\rho_f = \sup\{\rho > 0: \rho B_n \subset f(D)\}$$

where $\rho B_n = \{\rho z: z \in B_n\}$. Further we define

$$\rho(D) = \sup\{\rho_f: f \in \mathbb{S}(D)\}.$$

Obviously $\rho(D)$ is invariant under biholomorphic mappings. It is required to find the exact value of $\rho(D)$. Alexander [1] proved that for the unit polydisc U in $\mathbf{C}^n$, $\rho(U) = n^{-1/2}$.

In this paper we note that by appealing to the method of Alexander we are able to find the values $\rho(D)$ for the classical Cartan domains.

By a classical Cartan domain we understand a domain of one of the following four types:

$R_{\mathrm{I}}(r, s) = \{Z = (z_{jk}): I - Z\bar{Z}' > 0$, where Z is an $r \times s$ matrix$\}$ $(r \leq s)$,

$R_{\mathrm{II}}(p) = \{Z = (z_{jk}): I - Z\bar{Z}' > 0$, where Z is a symmetric matrix of order $p\}$,

$R_{\mathrm{III}}(q) = \{Z = (z_{jk}): I - Z\bar{Z}' > 0$, where Z is a skew-symmetric matrix of order $q\}$,

$R_{\mathrm{IV}}(n) = \{z = (z_1,\dots,z_n): 1 + |zz'|^2 - 2z\bar{z}' > 0,\ 1 - |zz'| > 0\}$.

Here I is the identity matrix and $\bar{Z}$ denotes the conjugate matrix of Z and Z' the transposed matrix of Z. The complex dimensions of these four domains are rs, $p(p+1)/2$, $q(q-1)/2$, n, respectively.

We shall prove the following theorems.

Received by the editors April 16, 1980.

1980 *Mathematics Subject Classification*. Primary 32H99.

Key words and phrases. Classical Cartan domain, holomorphic imbedding, bounded homogeneous domain, biholomorphic mapping.

THEOREM 1.
(1.1) $\rho(R_{\mathrm{I}}(r, s)) = r^{-1/2}$,
(1.2) $\rho(R_{\mathrm{II}}(p)) = p^{-1/2}$,
(1.3) $\rho(R_{\mathrm{III}}(q)) = [q/2]^{-1/2}$,
where $[q/2]$ *denotes the integral part of* $q/2$,
(1.4) $\rho(R_{\mathrm{IV}}(n)) = 2^{-1/2}$.

THEOREM 2. *If* $D_1,\dots,D_m$ *are classical Cartan domains, then*

$$\rho(D_1 \times \cdots \times D_m) = \left[\rho(D_1)^{-2} + \cdots + \rho(D_m)^{-2}\right]^{-1/2}.$$

2. We begin with two lemmas. By applying the method of Alexander [**1**] we are able to prove the following lemma. For the sake of completeness we give a proof.

LEMMA 1. *Let D be a bounded homogeneous domain in* $\mathbf{C}^n$ *satisfying the conditions*:
(1)

$$\left\{z = (z_1,\dots,z_n) \colon |z_{\alpha_j}| < 1 \textit{ for } j = 1,\dots,m \textit{ and } z_\alpha = 0 \textit{ for the other } \alpha\text{'s}\right\} \subset D,$$

where $1 \leqslant \alpha_1 < \cdots < \alpha_m \leqslant n$,
(2) *for each* j $(1 \leqslant j \leqslant m)$

$$\left\{z = (z_1,\dots,z_n) \colon |z_{\alpha_j}| = 1 \textit{ and } z_\alpha = 0 \textit{ for the other } \alpha\text{'s}\right\} \subset \partial D.$$

If $f \in \mathfrak{S}(D)$ *and* $\rho B_n \subset f(D)$, *then* $\rho \leqslant m^{-1/2}$.

PROOF. Let $f = (f_1,\dots,f_n)$ be a mapping in $\mathfrak{S}(D)$. Since D is homogeneous, we may assume that $f(0) = 0$. We set

$$g_\alpha(\zeta_1,\dots,\zeta_m) = f_\alpha(0,\dots,\zeta_1,\dots,\zeta_m,\dots,0) \qquad (\alpha = 1,\dots,n),$$

where ζ_j is in the α_jth position in the right-hand side. The function g_α is holomorphic in the polydisc $U = \{\zeta = (\zeta_1,\dots,\zeta_m) \colon |\zeta_j| < 1,\ j = 1,\dots,m\}$. Hence g_α admits an expansion

$$g_\alpha(\zeta) = \sum a^{(\alpha)}_{\nu_1\cdots\nu_m} \zeta_1^{\nu_1} \cdots \zeta_m^{\nu_m}, \qquad a^{(\alpha)}_{0\cdots 0} = 0,$$

which converges uniformly on compact subsets of U. Firstly, by condition (1) we have

$$\sum_{\alpha=1}^{n} \left(\sum \left|a^{(\alpha)}_{\nu_1\cdots\nu_m}\right|^2\right) = \lim_{r\to 1} \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{\sum_{\alpha=1}^{n} \left|g_\alpha\left(re^{i\theta_1},\dots,re^{i\theta_m}\right)\right|^2\right\} d\theta_1 \cdots d\theta_m \leqslant 1.$$

On the other hand, by condition (2), we have

$$\rho^2 \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left\{\sum_{\alpha=1}^{n} \left|g_\alpha(0,\dots,e^{i\theta_j},\dots,0)\right|^2\right\} d\theta_j = \sum_{\alpha=1}^{n} \left(\sum \left|a^{(\alpha)}_{0\cdots\nu_j\cdots 0}\right|^2\right),$$

where $g_\alpha(0,\ldots,e^{i\theta_j},\ldots,0)$ is the radial limit

$$\lim_{r\to 1} g_\alpha(0,\ldots,re^{i\theta_j},\ldots,0)$$

which exists a.e. on $\{\theta_j : 0 \leqslant \theta_j \leqslant 2\pi\}$. Therefore we obtain $m\rho^2 \leqslant 1$.

Next, instead of $R_{\mathrm{II}}(p)$ we consider the following modified domain:

$$\hat{R}_{\mathrm{II}}(p) = \left\{Z = (z_{jk}) : z_{jk} = \sqrt{2}\, x_{jk}(j \neq k),\ z_{jj} = x_{jj}, \text{ where } X = (x_{jk}) \in R_{\mathrm{II}}(p)\right\}.$$

Further, instead of $R_{\mathrm{IV}}(n)$ we consider the following domain:

$$R^*_{\mathrm{IV}}(n) = \Big\{z = (z_1,\ldots,z_n) : 1 + \left|z_1 z_2 + \tfrac{1}{2}\left(z_3^2 + \cdots + z_n^2\right)\right|^2$$
$$- \left(|z_1|^2 + \cdots + |z_n|^2\right) > 0,\ 1 - \left|z_1 z_2 + \tfrac{1}{2}\left(z_3^2 + \cdots + z_n^2\right)\right| > 0\Big\}.$$

The domain $R^*_{\mathrm{IV}}(n)$ is obtained from $R_{\mathrm{IV}}(n)$ by the biholomorphic mapping $\varphi = (\varphi_1,\ldots,\varphi_n)$, $\varphi_1(z) = z_1 + iz_2$, $\varphi_2(z) = z_1 - iz_2$, $\varphi_\alpha(z) = \sqrt{2}\, z_\alpha$ $(\alpha = 3,\ldots,n)$.

We have the following:

LEMMA 2.

$$B_n \subset R_{\mathrm{I}}(r,s) \subset \sqrt{r}\, B_n \qquad (n = rs),$$
$$B_n \subset \hat{R}_{\mathrm{II}}(p) \subset \sqrt{p}\, B_n \qquad (n = p(p+1)/2),$$
$$B_n \subset R_{\mathrm{III}}(q) \subset \sqrt{[q/2]}\, B_n \qquad (n = q(q-1)/2),$$
$$B_n \subset R^*_{\mathrm{IV}}(n) \subset \sqrt{2}\, B_n.$$

PROOF. In [4] we showed that

$$R_{\mathrm{I}}(r,s) \subset \sqrt{r}\, B_n,\quad \hat{R}_{\mathrm{II}}(p) \subset \sqrt{p}\, B_n,\quad R_{\mathrm{III}}(q) \subset \sqrt{[q/2]}\, B_n.$$

To show that $B_n \subset R_{\mathrm{I}}(r,s)$, let $\lambda_1,\ldots,\lambda_r$ be the characteristic roots of $Z\bar{Z}'$, where Z is an $r \times s$ matrix in B_n. Then we have

$$\lambda_1 + \cdots + \lambda_r = \operatorname{trace} Z\bar{Z}' = \sum_{j=1}^{r} \sum_{k=1}^{s} |z_{jk}|^2 < 1$$

and $\lambda_j \geqslant 0$ $(j = 1,\ldots,r)$; therefore $\lambda_j < 1$. This implies that $B_n \subset R_{\mathrm{I}}(r,s)$.

Similarly we obtain $B_n \subset \hat{R}_{\mathrm{II}}(p)$. If Z is a skew-symmetric matrix, the characteristic roots of $Z\bar{Z}'$ are double roots or equal to 0 [2]. Hence we have $B_n \subset R_{\mathrm{III}}(q)$. The last relation $B_n \subset R^*_{\mathrm{IV}}(n) \subset \sqrt{2}\, B_n$ is obvious.

3. We now turn to the proof of theorems. We consider an $r \times s$ matrix Z such that

$$Z = \begin{pmatrix} \zeta_1 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & \zeta_2 \cdots 0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots \zeta_r & 0 \cdots 0 \end{pmatrix}.$$

If $|\zeta_j| < 1$ $(j = 1,\ldots,r)$, then Z belongs to $R_{\mathrm{I}}(r,s)$. Further if $|\zeta_k| = 1$ and $\zeta_j = 0$ $(j \neq k)$, then Z belongs to $\partial R_{\mathrm{I}}(r,s)$. Therefore (1.1) is an immediate consequence of Lemmas 1 and 2. Similarly we obtain $\rho(\hat{R}_{\mathrm{II}}(p)) = p^{-1/2}$.

Considering a skew-symmetric matrix Z of order q such that

$$Z = \begin{pmatrix} 0 & \zeta_1 \\ -\zeta_1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & \zeta_m \\ -\zeta_m & 0 \end{pmatrix} \qquad (q = 2m)$$

or

$$Z = \begin{pmatrix} 0 & \zeta_1 \\ -\zeta_1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & \zeta_m \\ -\zeta_m & 0 \end{pmatrix} \dotplus 0 \qquad (q = 2m + 1),$$

we obtain (1.3). To prove (1.4) we consider a point z in $\mathbf{C}^n$ of the form $z = (\zeta_1, \zeta_2, 0, \ldots, 0)$, Then we have

$$\rho(R^*_{\mathrm{IV}}(n)) = 2^{-1/2}.$$

This completes the proof of Theorem 1.

Theorem 2 follows from Lemmas 1, 2 and the following fact: if $D_1, \ldots, D_m$ are domains satisfying

$$B_{n_\nu} \subset D_\nu \subset \rho_\nu B_{n_\nu} \qquad (\nu = 1, \ldots, m),$$

then

$$B_n \subset D_1 \times \cdots \times D_m \subset \sqrt{\rho_1^2 + \cdots + \rho_m^2}\, B_n \qquad (n = n_1 + \cdots + n_m).$$

References

1. H. Alexander, *Extremal holomorphic imbeddings between the ball and polydisc*, Proc. Amer. Math. Soc. **68** (1978), 200–202.

2. L. K. Hua, *On the theory of automorphic functions of a matrix variable*. I. *Geometrical basis*, Amer. J. Math. **66** (1944), 470–488.

3. ______, *Harmonic analysis of functions of several complex variables in the classical domains*, Transl. Math. Monographs, vol. 6, Amer. Math. Soc., Providence, R. I., 1963.

4. Y. Kubota, *An extremal problem on the classical Cartan domains*, Kodai Math. J. **4** (1981), 278–287.

5. K. H. Look, *Schwarz lemma and analytic invariants*, Sci. Sinica **7** (1958), 453–504.

Department of Mathematics, Tokyo Gakugei University, Koganei-shi, Tokyo 184, Japan