A NOTE ON HOLOMORPHIC IMBEDDINGS OF
THE CLASSICAL CARTAN DOMAINS
INTO THE UNIT BALL

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Abstract. Let D be a classical Cartan domain and let B be the unit ball. We find
the exact value of the supremum of the set of positive numbers ρ satisfying the
condition: ρB ⊂ f(D) for a certain holomorphic imbedding f: D → B.

1. This paper is concerned with the following problem: Let D be a bounded
homogeneous domain in C^n. By S(D) we denote the family of holomorphic
imbeddings f of D into the unit ball B_n in C^n such that f(D) ⊃ 0. For each f ∈ S(D)
we define

ρ_f = sup{ρ > 0: ρB_n ⊂ f(D)}

where ρB_n = {ρz: z ∈ B_n}. Further we define

ρ(D) = sup{ρ_f: f ∈ S(D)}.

Obviously ρ(D) is invariant under biholomorphic mappings. It is required to find
the exact value of ρ(D). Alexander [1] proved that for the unit polydisc U in C^n,
ρ(U) = n^{-1/2}.

In this paper we note that by appealing to the method of Alexander we are able to
find the values ρ(D) for the classical Cartan domains.

By a classical Cartan domain we understand a domain of one of the following
types:

R_1(r, s) = {Z = (z_{jk}): I - ZZ' > 0, where Z is an r × s matrix} (r ≤ s),
R_II(p) = {Z = (z_{jk}): I - ZZ' > 0, where Z is a symmetric matrix of order p},
R_III(q) = {Z = (z_{jk}): I - ZZ' > 0, where Z is a skew-symmetric matrix of order
q},
R_IV(n) = {z = (z_1, ..., z_n): 1 + |zz'|^2 - 2zz' > 0, 1 - |zz'| > 0}.

Here I is the identity matrix and Z denotes the conjugate matrix of Z and Z' the
transposed matrix of Z. The complex dimensions of these four domains are rs, 
p(p + 1)/2, q(q - 1)/2, n, respectively.

We shall prove the following theorems.

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Theorem 1.

(1.1) \( \rho(R_1(r, s)) = r^{-1/2} \),

(1.2) \( \rho(R_{11}(p)) = p^{-1/2} \),

(1.3) \( \rho(R_{11}(q)) = [q/2]^{-1/2} \),

where \([q/2]\) denotes the integral part of \(q/2\),

(1.4) \( \rho(R_{1v}(n)) = 2^{-1/2} \).

Theorem 2. If \( D_1, \ldots, D_m \) are classical Cartan domains, then

\[
\rho(D_1 \times \cdots \times D_m) = \left[ \rho(D_1)^{-2} + \cdots + \rho(D_m)^{-2} \right]^{-1/2}.
\]

2. We begin with two lemmas. By applying the method of Alexander [1] we are able to prove the following lemma. For the sake of completeness we give a proof.

Lemma 1. Let \( D \) be a bounded homogeneous domain in \( \mathbb{C}^n \) satisfying the conditions:

(1) \[
\{ z = (z_1, \ldots, z_n) : |z_{a_j}| < 1 \text{ for } j = 1, \ldots, m \text{ and } z_a = 0 \text{ for the other } a's \} \subset D,
\]

where \( 1 \leq a_1 < \cdots < a_m \leq n \),

(2) for each \( j (1 \leq j \leq m) \)

\[
\{ z = (z_1, \ldots, z_n) : |z_{a_j}| = 1 \text{ and } z_a = 0 \text{ for the other } a's \} \subset \partial D.
\]

If \( f \in \mathcal{S}(D) \) and \( \rho B_n \subset f(D) \), then \( \rho \leq m^{-1/2} \).

Proof. Let \( f = (f_1, \ldots, f_n) \) be a mapping in \( \mathcal{S}(D) \). Since \( D \) is homogeneous, we may assume that \( f(0) = 0 \). We set

\[
g_a(\zeta_1, \ldots, \zeta_m) = f_a(0, \ldots, \zeta_1, \ldots, \zeta_m, 0) \quad (\alpha = 1, \ldots, n),
\]

where \( \zeta_j \) is in the \( a_j \)th position in the right-hand side. The function \( g_a \) is holomorphic in the polydisc \( U = \{ \zeta = (\zeta_1, \ldots, \zeta_m) : |\zeta_j| < 1, j = 1, \ldots, m \} \). Hence \( g_a \) admits an expansion

\[
g_a(\zeta) = \sum \alpha \sum_{\nu_1, \ldots, \nu_m} a_{\alpha}^{(\nu_1, \ldots, \nu_m)} \zeta_{\nu_1} \cdots \zeta_{\nu_m}, \quad a_{0, \ldots, 0}^{(\alpha)} = 0,
\]

which converges uniformly on compact subsets of \( U \). Firstly, by condition (1) we have

\[
\sum_{\alpha=1}^{n} \left( \sum_{\nu_1, \ldots, \nu_m} |a_{\alpha}^{(\nu_1, \ldots, \nu_m)}|^2 \right) = \lim_{r \to 1} \frac{1}{(2 \pi)^m} \int_{0}^{2 \pi} \cdots \int_{0}^{2 \pi} \left\{ \sum_{\alpha=1}^{n} \left| g_a(re^{i \theta_1}, \ldots, re^{i \theta_m}) \right|^2 \right\} d \theta_1 \cdots d \theta_m \leq 1.
\]

On the other hand, by condition (2), we have

\[
\rho^2 \leq \frac{1}{2 \pi} \int_{0}^{2 \pi} \left\{ \sum_{\alpha=1}^{n} |g_a(0, \ldots, e^{i \theta_j}, \ldots, 0)|^2 \right\} d \theta_j = \sum_{\alpha=1}^{n} \left( \sum |a_{0, \ldots, 0}^{(\alpha)}|^2 \right),
\]
where \( g_n(0, \ldots, re^{i\theta}, \ldots, 0) \) is the radial limit
\[
\lim_{r \to 1} g_n(0, \ldots, re^{i\theta}, \ldots, 0)
\]
which exists a.e. on \( (\theta_j: 0 < \theta_j < 2\pi) \). Therefore we obtain \( m_\rho^2 \leq 1 \).

Next, instead of \( R_{11}(p) \) we consider the following modified domain:
\[
\hat{R}_{11}(p) = \left\{ Z = (z_{jk}): z_{jk} = \sqrt{2} x_{jk} (j \neq k), z_{jj} = x_{jj}, \text{where } X = (x_{jk}) \in R_{11}(p) \right\}.
\]
Further, instead of \( R_{11}(n) \) we consider the following domain:
\[
R_{11}^*(n) = \left\{ z = (z_1, \ldots, z_n): 1 + \left| z_1 z_2 + \frac{1}{2} (z_3^2 + \cdots + z_n^2) \right|^2 - \left( |z_1|^2 + \cdots + |z_n|^2 \right) > 0, 1 - |z_1 z_2 + \frac{1}{2} (z_3^2 + \cdots + z_n^2)| > 0 \right\}.
\]
The domain \( R_{11}^*(n) \) is obtained from \( R_{11}(n) \) by the biholomorphic mapping
\[
\varphi = (\varphi_1, \ldots, \varphi_n), \quad \varphi(z) = z_1 + iz_2, \quad \varphi_2(z) = z_1 - iz_2, \quad \varphi_n(z) = \sqrt{2} z_n (n = 3, \ldots, n).
\]
We have the following:

**Lemma 2.**
\[
B_n \subset R_1(r, s) \subset \sqrt{r} B_n \quad (n = rs),
\]
\[
B_n \subset \hat{R}_{11}(p) \subset \sqrt{p} B_n \quad (n = p(p + 1)/2),
\]
\[
B_n \subset R_{111}(q) \subset \sqrt{[q/2]} B_n \quad (n = q(q - 1)/2),
\]
\[
B_n \subset R_{11}^*(n) \subset \sqrt{2} B_n.
\]

**Proof.** In [4] we showed that
\[
R_1(r, s) \subset \sqrt{r} B_n, \quad \hat{R}_{11}(p) \subset \sqrt{p} B_n, \quad R_{111}(q) \subset \sqrt{[q/2]} B_n.
\]
To show that \( B_n \subset R_1(r, s) \), let \( \lambda_1, \ldots, \lambda_r \) be the characteristic roots of \( ZZ' \), where \( Z \) is an \( r \times s \) matrix in \( B_n \). Then we have
\[
\lambda_1 + \cdots + \lambda_r = \text{trace } ZZ' = \sum_{j=1}^r \sum_{k=1}^s |z_{jk}|^2 < 1
\]
and \( \lambda_j \geq 0 (j = 1, \ldots, r) \); therefore \( \lambda_j < 1 \). This implies that \( B_n \subset R_1(r, s) \).

Similarly we obtain \( B_n \subset \hat{R}_{11}(p) \). If \( Z \) is a skew-symmetric matrix, the characteristic roots of \( ZZ' \) are double roots or equal to 0 [2]. Hence we have \( B_n \subset R_{111}(q) \). The last relation \( B_n \subset R_{11}^*(n) \subset \sqrt{2} B_n \) is obvious.

3. We now turn to the proof of theorems. We consider an \( r \times s \) matrix \( Z \) such that
\[
Z = \begin{pmatrix}
\xi_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \xi_2 & 0 & \cdots & 0 & \cdots & 0 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \xi_r & 0 & \cdots & 0
\end{pmatrix}
\]
If \( |\xi_j| < 1 (j = 1, \ldots, r) \), then \( Z \) belongs to \( R_1(r, s) \). Further if \( |\xi_k| = 1 \) and \( \xi_j = 0 \) \( (j \neq k) \), then \( Z \) belongs to \( \partial R_1(r, s) \). Therefore (1.1) is an immediate consequence of Lemmas 1 and 2. Similarly we obtain \( R_{11}(p) = p^{-1/2} \).
Considering a skew-symmetric matrix $Z$ of order $q$ such that

$$Z = \left( \begin{array}{cc} 0 & \xi_1 \\ -\xi_1 & 0 \end{array} \right) + \cdots + \left( \begin{array}{cc} 0 & \xi_m \\ -\xi_m & 0 \end{array} \right) \quad (q = 2m)$$

or

$$Z = \left( \begin{array}{cc} 0 & \xi_1 \\ -\xi_1 & 0 \end{array} \right) + \cdots + \left( \begin{array}{cc} 0 & \xi_m \\ -\xi_m & 0 \end{array} \right) + 0 \quad (q = 2m + 1),$$

we obtain (1.3). To prove (1.4) we consider a point $z$ in $\mathbb{C}^n$ of the form $z = (\xi_1, \xi_2, 0, \ldots, 0)$, Then we have

$$\rho(R_v^*(n)) = 2^{-1/2}.$$ 

This completes the proof of Theorem 1.

Theorem 2 follows from Lemmas 1, 2 and the following fact: if $D_1, \ldots, D_m$ are domains satisfying

$$B_n \subset D_v \subset \rho_v B_{n_v} \quad (v = 1, \ldots, m),$$

then

$$B_n \subset D_1 \times \cdots \times D_m \subset \sqrt{\rho_1^2 + \cdots + \rho_m^2} B_n \quad (n = n_1 + \cdots + n_m).$$

**References**


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