PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

OSCILLATION CRITERIA FOR THE SUBLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this paper, we propose an oscillation theorem to the sublinear Emden-Fowler equation $\Delta u + g(x)u^{\gamma} = 0$ for $n = 2$. By using this, we will answer to the open problem of Noussair and Swanson in the case of bounded solutions.

1. Introduction. Recently, Noussair and Swanson [**6, 7, 8**] have developed some oscillation criteria for the Schrödinger equation. We also refer to Kreith and Travis [**4**] and others [**1, 3, 5**] for related results. Especially, Noussair and Swanson [**7**] get a necessary and sufficient condition for nonoscillation of a sublinear Emden-Fowler equation in the case of $n \geq 3$. But, the important case of $n = 2$ has not been settled up to date. In this paper, we propose an oscillation theorem for the case of $n = 2$.

The partial differential equations to be examined are of the form

$$\Delta u + \sum_{i=1}^{n} g_i(x) |u|^{\gamma_i} \operatorname{sgn} u = 0, \tag{1}$$
$$x \in \Omega, 0 < \gamma_1 < 1 \text{ and } \gamma_i > 0 \text{ for } i = 2, 3, \ldots, n,$$

where Ω is an exterior domain $\Omega \subset R^2$ and $g_i(x) \in C(\Omega)$ for $i = 1, 2, \ldots, n$. Equation (1) is said to be oscillatory in Ω if every solution $u(x) \in C^2(\Omega)$ of (1) that is nontrivial in any neighbourhood of infinity has arbitrarily large zeros: that is, the set $\{x \in \Omega: u(x) = 0\}$ is unbounded. Equation (1) is called nonoscillatory in an unbounded domain Ω whenever (1) has a one-signed solution in $\Omega \cap G_r$ for some positive number r. Let $|x|$ denote the Euclidean norm of a point $x = (x_1, x_2)$ in a real 2-dimensional Euclidean space R^2. Define

$$S_r = \{x \in R^2: |x| = r\}, \quad G_r = \{x \in R^2: |x| > r\}$$

and

$$G(r, s) = \{x \in R^2: r < |x| < s\}, \qquad 0 < r < s.$$

We define a function $f(r)$ on $(0, \infty)$ by the equation

$$\bar{u}(r) = f(r) = \frac{1}{|S_r|} \int_{S_r} u(x)\, dS, \qquad |S_r| = 2\pi r,$$

which is a spherical mean of the function $u(x)$, and we define a function $\hat{u}(r) = \sup_{|x|=r} u(x)$.

Received by the editors May 1, 1981.
1980 *Mathematics Subject Classification*. Primary 35B05.
Key words and phrases. Schrödinger equation, oscillatory, nonoscillatory.

2. The results.

THEOREM 1. *Consider the equation*

$$\Delta u + \sum_{i=1}^{n} g_i(x)\,|\,u\,|^{\gamma_i}\operatorname{sgn} u = 0 \quad \text{in } G_a, \tag{2}$$

where $0 < \gamma_1 < 1$, $\gamma_i > 0$ *and* $g_i(x) \geqslant 0$ *in* G_a *for* $i = 2, 3, \ldots, n$. *If*

$$\int_{r_0}^{\infty} r(\log r)^{\gamma_1}\bar{g}_1(r)\,dr = +\infty \quad \text{for some } r_0 > 0, \tag{3}$$

then every bounded solution of (2) *is oscillatory.*

PROOF. Suppose to the contrary that there exists a solution $u(x)$ of (2) which has no zero in G_b for some $b > a > 0$. We may assume that $u(x) > 0$ in G_b, since a parallel argument holds if $u(x) < 0$ in G_b. We write the γ_1 in (2) simply by γ. We may take a constant k such that

$$0 < \gamma < k < 1 \quad \text{and} \quad 1 - k < \gamma. \tag{4}$$

By the hypothesis that $g_i(x) \geqslant 0$ in G_b, we have $\Delta u \leqslant 0$ in Ω_b. As $u(x)$ is bounded, it follows that

$$\bar{u}(t) \leqslant c \quad \text{for } t \geqslant T_1 > b \text{ and a constant } c > 0. \tag{5}$$

Put $v(x) = (\log|\,x\,|)^{\gamma}$, $w = u^{1-k}$. By using the following Green's formula

$$\int_{G(T_1,t)} (v\Delta w - w\Delta v)\,dx = \int_{\partial G(T_1,t)} \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n} \right) dS,$$

we obtain

$$\begin{aligned}\int_{T_1}^{t} &\left(r(\log r)^{\gamma}\overline{\Delta w} + \gamma(1-\gamma)\frac{\bar{w}}{r(\log r)^{2-\gamma}} \right) dr \\ &= t(\log t)^{\gamma}\bar{w}'(t) - \gamma(\log t)^{\gamma-1}\bar{w}(t) + c_1 \\ &= t(\log t)^{2\gamma}\frac{d}{dt}\left[\frac{\bar{w}(t)}{(\log t)^{\gamma}} \right] + c_1 \quad \text{where } c_1 \text{ is a constant.}\end{aligned} \tag{6}$$

It follows from (4) and (5) that

$$\bar{w} = \overline{u^{1-k}} \leqslant (\bar{u})^{1-k} \leqslant c^{1-k} \equiv c_2 \quad \text{for } t \geqslant T_1. \tag{7}$$

By using (7), we have

$$\begin{aligned} 0 < \int_{T_1}^{\infty} \frac{\bar{w}}{r(\log r)^{2-\gamma}}\,dr &\leqslant c_2 \int_{T_1}^{\infty} \frac{dr}{r(\log r)^{2-\gamma}} \\ &= \frac{c_2}{1-\gamma}(\log T_1)^{\gamma-1} < +\infty. \end{aligned} \tag{8}$$

We compute

$$\Delta w = (1-k)\frac{\Delta u}{u^k} - k(1-k)\frac{|\,\nabla u\,|^2}{u^{1+k}} \leqslant \frac{(1-k)(-g_1(x)u^{\gamma})}{u^k} \leqslant -c_3 g_1(x), \tag{9}$$

where c_3 is a positive constant and $\gamma - k$ is negative, and we used the boundedness of $u(x)$. From (9) and (3), we deduce that the left-hand integral of (6) tends to $-\infty$ as $t \to \infty$. By using (8) and (9), we obtain

$$-1 \geqslant t(\log t)^{2\gamma} \frac{d}{dt}\left[\frac{\bar{w}(t)}{(\log t)^{\gamma}}\right] \quad \text{for } t \geqslant T_2 \geqslant T_1. \tag{10}$$

By integrating (10) over $[T, t]$, for $T > \max\{T_2, e\}$, we have

$$\begin{aligned} &\frac{\bar{w}(t)}{(\log t)^{\gamma}} - \frac{\bar{w}(T)}{(\log T)^{\gamma}} \leqslant -\int_T^t \frac{dr}{r(\log r)^{2\gamma}} \\ &\qquad \begin{cases} = \dfrac{-1}{1-2\gamma}\left\{(\log t)^{1-2\gamma} - (\log T)^{1-2\gamma}\right\} & \text{for } 1-2\gamma \neq 0, \\ = -\log(\log t) + \log(\log T) & \text{for } 1-2\gamma = 0. \end{cases} \end{aligned} \tag{11}$$

Let $t \to \infty$ in (11). If $1 - 2\gamma \geqslant 0$, then we have an immediate contradiction to the boundedness of $\bar{w}(t)$. If $1 - 2\gamma < 0$, then (11) implies $\bar{w}(T) \geqslant c_4(\log T)^{1-\gamma}$ for sufficiently large T and a constant c_4. Since $1 - \gamma > 0$, we again have a contradiction. Q.E.D.

A special case of Theorem 1 is the following

COROLLARY. *Consider the sublinear equation*

$$\Delta u + g(x)\,|u|^{\gamma} \operatorname{sgn} u = 0 \quad \textit{in } G_a \textit{ where } 0 < \gamma < 1 \textit{ and } g(x) \geqslant 0 \textit{ in } G_a. \tag{12}$$

If

$$\int_{r_0}^{\infty} r(\log r)^{\gamma} \bar{g}(r)\, dr = +\infty \quad \textit{for some } r_0 > 0, \tag{13}$$

then every bounded solution of (12) *is oscillatory.*

THEOREM 2 [7, THEOREM 5.2]. *Suppose*

(i) $g(x) \geqslant 0$ *for all* $x \in \Omega$, $g \in C^{\alpha}(\overline{M})$ *for every bounded subdomain* $M \subset \Omega$, *and* $\hat{g} \in C^{\alpha}[a, b]$ *for some* $a > 0$, $0 < \alpha < 1$ *and for all* $b > a$.

The sublinear Emden-Fowler equation (12) *is nonoscillatory in an exterior domain in* R^2 *if*

$$\int_{r_0}^{\infty} r(\log r)^{\gamma} \hat{g}(r)\, dr < \infty \quad \textit{for some } r_0 > 0. \tag{14}$$

Our extra hypothesis is

$$\liminf_{r \to \infty} \bar{g}(r)/\hat{g}(r) > 0. \tag{15}$$

THEOREM 3. *If* $g(x)$ *satisfies* (15) *and the hypothesis of Theorem* 2, *then* (13) *is a necessary and sufficient condition for all bounded solutions of the sublinear Emden-Fowler equation* (12) *to be oscillatory in an exterior domain* $\Omega \subset R^2$.

PROOF. The sufficiency is contained in the Corollary. If (13) fails, then by (15) there exist positive numbers ε and r_0 such that

$$+\infty > \int_{r_0}^{\infty} r(\log r)^{\gamma} \bar{g}(r)\, dr > \varepsilon \int_{r_0}^{\infty} r(\log r)^{\gamma} \hat{g}(r)\, dr,$$

whence all solutions of (12) are nonoscillatory by Theorem 2. Q.E.D.

EXAMPLE. Consider the equation

$$\Delta u + \frac{1}{r^2(\log r)^{7/6}} u^{1/3} = 0. \tag{16}$$

Since

$$\int^{\infty} r(\log r)^{1/3} \frac{1}{r^2(\log r)^{7/6}}\, dr = \int^{\infty} \frac{1}{r(\log r)^{5/6}}\, dr = \int^{\infty} X^{-5/6}\, dX = +\infty, \tag{17}$$

every bounded solution of (16) is oscillatory by Theorem 1.

REMARK. We compute

$$\int^{\infty} r \cdot \frac{1}{r^2(\log r)^{7/6}}\, dr = \int^{\infty} \frac{dr}{r(\log r)^{7/6}} = \int^{\infty} X^{-7/6}\, dX < +\infty. \tag{18}$$

Equations (17) and (18) show that equation (16) is not contained in the results of Kitamura and Kusano [2] but is covered by Theorem 1.

ACKNOWLEDGEMENT. The author wishes to thank the referee for several helpful comments.

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