PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

BERNSTEIN'S THEOREM FOR THE POLYDISC

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ABSTRACT. A well-known theorem of Bernstein states that if a polynomial of degree N of a complex variable has its modulus no larger than one on the unit disc then the modulus of its derivative will not exceed N on the unit disc. The result has been extended to the case of polynomials on the unit ball in several complex variables. Here we generalize the theorem to the cases of the unit polydisc and the unit polycylinder which is a topological product of a unit ball and a unit polydisc.

Let $\mathbf{C}$ be the field of complex numbers and $\mathbf{C}^m$ the vector space over $\mathbf{C}$ of m-tuples. The unit ball B_m is the set of points $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$ such that $\|z\|^2 = |z_1|^2 + \cdots + |z_m|^2 < 1$ and $B_1 \equiv U$ is the open unit disc in $\mathbf{C}$. The unit polydisc is the cartesian product of m copies of U. Thus, for $z \in U^m$, we denote $\|z\| = \max_{1 \leq i \leq m} |z_i| < 1$. The distinguished boundary **[1, 2]** of the polydisc U^m is the torus $T^m = \{z\colon |z_i| = 1, i = 1, \ldots, m\}$. It is known that the maximum modulus of a function regular in the closure of a domain is assumed on the distinguished boundary of the domain. In particular, a polynomial on $\overline{U}^m$, the closure of the unit polydisc U^m of dimension m, assumes its maximum modulus on the torus T^m. In the case of the unit ball, the distinguished boundary is not a proper subset of the boundary which is the unit hypersphere $S_m = \{z\colon |z_1|^2 + \cdots + |z_m|^2 = 1\}$.

Let P be a polynomial of degree N in $z \in \mathbf{C}^m$. We denote by $D_i P$ the derivative of P with respect to z_i and define a directional derivative of P at z in the direction of u by

$$D_u P(z) = \sum_{i=1}^{m} u_i D_i P(z),$$

where $u = (u_1, \ldots, u_m)$ is an element in S_m or T^m with $\|z\| = 1$. We also denote

$$DP(z) = (D_1 P(z), \ldots, D_m P(z))$$

and

$$\|DP(z)\|^2 = \sum_{i=1}^{m} |D_i P(z)|^2.$$

With these notations and definitions, we now may state the extension of Bernstein's theorem in the unit ball obtained in [3] as follows.

Received by the editors April 20, 1981 and, in revised form, August 15, 1981.
1980 *Mathematics Subject Classification*. Primary 32A30, 30C10.
Key words and phrases. Bernstein's theorem, polynomial, polydisc, polycylinder.

THEOREM 1. *Let P be a polynomial of degree N with $|P(z)| \leq 1$ for z in the closure of the unit ball $\bar{B}_m = \{z\colon \|z\| \leq 1\}$. Then, for any $u \in S_m$, we have*

$$|D_u P(z)| \leq \|DP(z)\| \leq N.$$

This result is best possible and the equalities hold when P is a homogeneous polynomial of degree N with $u = z = z_0$ and $|P(z_0)| = 1$.

Here, in this paper, we obtain Bernstein's theorem for the polydisc and the polycylinder. If P is a polynomial of degree N in $z \in \mathbf{C}^m$ and $|P(z)| \leq 1$ for $z \in \bar{U}^m$, it is easy to see that $\|DP(z)\| \leq \sqrt{m}\,N$ by applying Bernstein's theorem on $U \subset \mathbf{C}$ for each $D_i P(z)$, $i = 1, \ldots, m$. We will obtain a better and generalized result for the inequality and extend the result to the polycylinder.

For any $z, u \in \bar{U}^m \subset \mathbf{C}^m$ and $s, t \in \mathbf{C}$, we know that $sz + tu \in \mathbf{C}^m$ and recall our definition

$$\|sz + tu\| = \max_{1 \leq i \leq m} |sz_i + tu_i|.$$

We define the set

$$V = \left\{(s, t) \in \mathbf{C}^2\colon \|sz + tu\| \leq 1 \text{ for all } z, u \in \bar{U}^m\right\}$$

and obtain the following lemmas.

LEMMA 1. *The set V is a subset of the closure of the 2-ball B_2, that is, $V \subseteq \bar{B}_2$.*

PROOF. We define the set $V^* = \{(s, t) \in \mathbf{C}^2\colon |s| + |t| \leq 1\}$. For any given $(s, t) \in V$, we can choose some $z_1, u_1 \in T \equiv T^1$ such that $\arg sz_1 = \arg tu_1$. Hence

$$|s| + |t| = |sz_1| + |tu_1| = |sz_1 + tu_1| \leq \|sz + tu\| \leq 1,$$

where z_1 and u_1 are the first coordinates of z and u in $\bar{U}^m$ repsectively. Thus, we have $V \subseteq V^*$. On the other hand, from the definition

$$\|sz + tu\| = \max_{1 \leq i \leq m} |sz_i + tu_i| \leq |s| + |t| \leq 1 \quad \text{for } z, u \in \bar{U}^m,$$

we have $V^* \subseteq V$. Therefore, the equality $V = V^*$ together with the obvious relation $V^* \subseteq \bar{B}_2$ implies the result $V \subseteq \bar{B}_2$.

LEMMA 2. *Let Q be a polynomial of degree N in s and t, and $|Q(s, t)| \leq 1$ for $(s, t) \in V$. Then, for every $(v, w) \in S_2$, we have $|D_{(v,w)} Q(s, t)| \leq N$.*

PROOF. From $V \subseteq \bar{B}_2$ in Lemma 1 and an application of Theorem 1 for the unit ball $\bar{B}_2$, we have

$$|D_{(v,w)} Q(s, t)| \leq \max_{(x, y) \in \bar{B}_2} |D_{(v,w)} Q(x, y)| \leq N.$$

Now we are ready to establish Bernstein's theorem for a polynomial on a polydisc as follows.

THEOREM 2. *Let P be a polynomial of degree N in $\mathbf{C}^m$ and $|P(z)| \leq 1$ for $z \in \bar{U}^m$. For any $u \in T^m$, we have*

$$m^{-1/2} |D_u P(z)| \leq \|DP(z)\| \leq N.$$

PROOF. Let $Q(s, t) = P(sz + tu)$ for $z, u \in \bar{U}^m$ and $(s, t) \in V$. Hence, $Q(s, t)$ is a polynomial of degree N in s and t. For any $(v, w) \in S_2$, Lemma 2 gives

$$\left|v\frac{\partial Q}{\partial s}(s, t) + w\frac{\partial Q}{\partial t}(s, t)\right| = |D_{(v,w)}Q(s, t)| \leqslant N.$$

Here, we notice that

$$\frac{\partial Q}{\partial t}(s, t) = \sum_{i=1}^{m} u_i D_i P(sz + tu).$$

In particular, by choosing $s = w = 1$, $t = v = 0$, we have

$$N \geqslant \left|\frac{\partial Q}{\partial t}(1,0)\right| = \left|\sum_{i=1}^{m} u_i D_i P(z)\right| = |D_u P(z)|$$

for any $u \in T^m \subseteq \bar{U}^m$. On the other hand, from the Schwarz inequality

$$|D_u P(z)| = \left|\sum_{i=1}^{m} u_i D_i P(z)\right| \leqslant \sqrt{\Sigma_{i=1}^m |u_i|^2}\sqrt{\Sigma_{i=1}^m |D_i P(z)|^2} = \sqrt{m}\,\|DP(z)\|$$

and therefore, we have

$$m^{-1/2}|D_u P(z)| \leqslant \|DP(z)\| \leqslant \sum_{i=1}^{m} |D_i P(z)| = \max_{u} |D_u P(z)| \leqslant N.$$

This proves the theorem.

From the last inequality in the proof of the theorem, we obtain the following result which is a refinement of Theorem 2.

COROLLARY 1. *Let P be a polynomial of degree N in $\mathbf{C}^m$ and $|P(z)| \leqslant 1$ for $z \in \bar{U}^m$. Then*

$$\|DP(z)\| \leqslant \sum_{i=1}^{m} |D_i P(z)| \leqslant N.$$

Next, we consider Bernstein's theorem on a polycylinder.

THEOREM 3. *Let P be a polynomial of degree N in $\mathbf{C}^{m+1}$ and $|P(z)| \leqslant 1$ for $z = (z_0, z_1, \ldots, z_m) \in \bar{U} \times \bar{B}_m$. For any $u = (u_0, u_1, \ldots, u_m) \in T \times S_m$, we have*

$$2^{-1/2}|D_u P(z)| \leqslant \|DP(z)\| \leqslant N.$$

PROOF. As in the proof of Theorem 2, if we set $Q(s, t) = P(sz + tu)$ for $z, u \in \bar{U} \times \bar{B}_m$ and $(s, t) \in V = V^*$, then

$$N \geqslant \left|\frac{\partial Q}{\partial t}(1,0)\right| = \left|\sum_{i=0}^{m} u_i D_i P(z)\right| = |D_u P(z)|$$

for any $z \in \bar{U} \times \bar{B}_m$ and $u \in T \times S_m$. Hence $|u_0| = 1$ and $\Sigma_{i=1}^m |u_i|^2 = 1$, and

$$|D_u P(z)| = \left|\sum_{i=0}^{m} u_i D_i P(z)\right| \leqslant \sqrt{\Sigma_{i=0}^m |u_i|^2}\sqrt{\Sigma_{i=0}^m |D_i P(z)|^2} = \sqrt{2}\,\|DP(z)\|.$$

Furthermore,

$$N \geqslant \max_u \left| \sum_{i=0}^{m} u_i D_i P(z) \right| = |D_0 P(z)| + \max_u \left| \sum_{i=1}^{m} u_i D_i P(z) \right|$$

$$= |D_0 P(z)| + \sqrt{\textstyle\sum_{i=1}^{m} |D_i P(z)|^2} \geqslant \|DP(z)\| \geqslant 2^{-1/2} |D_u P(z)|$$

and the theorem is proved.

From the above proof, it is easy to see that we can obtain the following form of Bernstein's theorem for polynomials on a generalized polycylinder.

COROLLARY 2. *Let P be a polynomial of degree N and $|P(z)| \leqslant 1$ for $z \in \overline{U}^m \times \overline{B}_{m_1} \times \cdots \times \overline{B}_{m_n}$. Then, for any $u \in T^m \times S_{m_1} \times \cdots \times S_{m_n}$, we have*

$$(m+n)^{-1/2} |D_u P(z)| \leqslant \|DP(z)\| \leqslant N.$$

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