PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

PEAK SETS FOR THE REAL PART OF A FUNCTION ALGEBRA

EGGERT BRIEM

ABSTRACT. We show that if A is a function algebra with the property that every peak set for $\operatorname{re} A$ is an interpolation set for A then $A = C(X)$.

1. Introduction. By realizing a function algebra as the space of all real-valued continuous affine functions on a certain compact convex set (see details below), one can obtain information about the algebra using results from the theory of compact convex sets (see e.g. [**2**, **4**, **5**, **6** and **8**]). One result states that if A is a function algebra on a compact Hausdorff space X with the property that each peak set for $\operatorname{re} A$ is a peak set for A, then $A = C(X)$ [**1**, Theorem 5.9.5]. We give a short proof of this result using a recent characterization of simplexes, and we also show that $A = C(X)$ if each peak set for $\operatorname{re} A$ is an interpolation set for A. This last result may be viewed as an extension of a theorem of Glicksberg [**7**], which states that $A = C(X)$ if each closed subset of X is an interpolation set for A.

2. The notation, terminology and basic results can be found in the book of Asimow and Ellis [**1**]. In what follows A is a function algebra on a compact Hausdorff space X with state space S_A and Z is the complex state space, i.e. the w^*-compact convex subset of A^* given by $Z = \operatorname{co}(S_A \cup -iS_A)$. The embedding $a \to \theta(a)$ where $\theta(a)(p) = \operatorname{re} p(a)$ for all $p \in A$ is a bicontinuous real-linear isomorphism of A onto $A(Z)$, the Banach space of all continuous real-valued affine functions on Z.

PROPOSITION 1. *The following conditions are equivalent*:
(i) *each peak face of Z is a parallel face*,
(ii) *Z is a Choquet simplex*,
(iii) $A = C(X)$.

PROOF. (i) $\Leftrightarrow$ (ii) see [**3**, Theorem 2], and (ii) $\Leftrightarrow$ (iii) see [**9**, Theorem 2 and Corollary 3.5].

PROPOSITION 2. *If F is an interpolation set for A and a peak set for* $\operatorname{re} A$, *then F is a peak set for A.*

Received by the editors March 30, 1981 and, in revised form, September 21, 1981.
1980 *Mathematics Subject Classification*. Primary 46J10, 46A55.
Key words and phrases. Function algebra, peak set, interpolation set, compact convex set, simplex.

PROOF. Let $a = u + iv \in A$ where u and v are real functions be such that $u = 0$ on F and $u < 0$ on $X \setminus F$. The function $a_1 = \exp(a)$ belongs to A, $|a_1| = 1$ on F and $|a_1| < 1$ on $X \setminus F$. Since the space of restrictions $A|_F$ is closed, the open mapping theorem implies the existence of a constant C such that each $f \in A|_F$ has an extension $a \in A$ with $\|a\| \leq C\|f\|_F$. In particular there exists, for each natural number n, a function $b_n \in A$ such that $b_n = \exp(-na)$ on F and such that $\|b_n\| \leq C$. Thus the sequence $\{a_1^n b_n\}$ which belongs to A has the following properties: $\|a_1^n b_n\| \leq C$, $a_1^n b_n = 1$ on F and $\lim a_1^n b_n = 0$ uniformly on each compact subset of $X \setminus F$. Further, F is a G_δ-set since it is a peak set for $\operatorname{re} A$. It follows that F is a peak set for A.

THEOREM 3. *Let A be a function algebra on a compact Hausdorff space X. Then the following conditions are equivalent*:

(i) *each peak set for* re A *is an interpolation set for* A,

(ii) *each peak set for* re A *is a peak set for* A,

(iii) $A = C(X)$,

(iv) *each closed* $E \subset X$ *is an interpolation set for* A.

PROOF. (i) $\Leftrightarrow$ (ii) is Proposition 2. (ii) $\Leftrightarrow$ (iii): Let F be a peak face of Z. Then $E_1 = F \cap X$ and $E_2 = i(F \cap -X)$ are peak sets for $\operatorname{re} A$ and hence for A (we are thinking of X as a subset of S_A via the natural homeomorphism which takes each $x \in X$ to the evaluation functional at x), and thus $F_1 = \overline{\text{co}}(E_1 \cup -iE_1)$ and $F_2 = \overline{\text{co}}(E_2 \cup -iE_2)$ are split faces of Z [**1**, Theorem 4.7.1]. If $\mu \in \partial A(Z)^\perp$ then $\mu|_{F_1}$ and $\mu|_{F_2}$ both are in $A(Z)^\perp$. Since the function which is 1 on S_A and 0 on $-iS_A$ is in $A(Z)$, it follows that $\mu(E_1) = \mu(-iE_1) = 0$ and that $\mu(E_2) = \mu(-iE_2) = 0$ and hence that $\mu(F) = 0$. Thus F is a parallel face. Since we have proved that each peak face of Z is a parallel face, Proposition 1 shows that $A = C(X)$. The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are trivial.

REFERENCES

1. L. Asimow and A. J. Ellis, *Convexity theory and its applications in functional analysis*, LMS Monographs, No. 16, Academic Press, New York, 1980.

2. E. Briem, *Split faces associated with function algebras*, J. London Math. Soc. (2) **9** (1975), 446–450.

3. ______, *A characterization of simplexes by parallel faces*, Bull. London Math. Soc. **12** (1980), 55–59.

4. A. J. Ellis, *On split faces and function algebras*, Math. Ann. **195** (1972), 159–166.

5. ______, *On facially continuous functions in function algebras*, J. London Math. Soc. (2) **5** (1972), 561–564.

6. ______, *Central decompositions and the essential set for the space A(K)*, Proc. London Math. Soc. (3) **26** (1973), 564–576.

7. I. Glicksberg, *Function algebras with closed restrictions*, Proc. Amer. Math. Soc. **14** (1963), 158–161.

8. B. Hirsberg, *M-ideals in function spaces and algebras*, Israel J. Math. **12** (1972), 133–146.

9. B. Hirsberg and A. J. Lazar, *Complex Lindenstrauss spaces with extreme points*, Trans. Amer. Math. Soc. **186** (1973), 141–150.

RAUNVISINDASTOFNUN HÁSKÓLANS, SCIENCE INSTITUTE, UNIVERSITY OF ICELAND, DUNHAGA 3, REYKJAVIK, ICELAND