PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

LIE PRODUCTS, CLOSED GEODESICS AND FUCHSIAN GROUPS

ALAN F. BEARDON

ABSTRACT. Two results concerning closed geodesics on a Riemann surface are proved. The method of proof indirectly involves the Lie product of two complex matrices.

1. Introduction. In this paper we extend the discussion given in [**1**] in which the Lie product $AB - BA$ of A and B in $SL(2, C)$ is used to derive a result on geodesics on a Riemann surface. The essential idea in [**1**] is to relate the Lie product to the geometry of hyperbolic three-space and then consider the given Riemann surface as the quotient of a hyperbolic plane section by an appropriate discrete group of isometries.

We shall relate the Lie product of A and B to the classical factorisation of A and B in terms of other isometries of order two and by doing so we reveal that the Theorem in [**1**] is merely one of a sequence of similar results.

2. Isometries of the hyperbolic plane. Let Δ denote the hyperbolic plane with ρ the hyperbolic metric; for the moment, we regard Δ as being embedded in the extended complex plane and the isometries of Δ are then Möbius transformations of z or $\bar{z}$. Each conformal isometry is either elliptic, parabolic or hyperbolic.

If A is elliptic, then A has a unique fixed point w in Δ and A is a hyperbolic rotation of angle 2θ, say, about w. Note that A can be expressed as $A = \sigma_2\sigma_1$ where σ_j is the reflection in a geodesic L_j passing through w. Either L_1 or L_2 can be chosen arbitrarily; the remaining L_j is uniquely determined and L_1 and L_2 intersect at w at an angle θ.

If A is hyperbolic, then A fixes two distinct points on the boundary of Δ and the hyperbolic geodesic $\mathcal{L}_A$ joining these points is called the axis of A. Note that $\mathcal{L}_A$ is the unique A-invariant geodesic in Δ. The displacement $\rho(z, Az)$, $z \in \Delta$, is minimum when z is on $\mathcal{L}_A$ and then the displacement is the translation length T_A of A.

Any hyperbolic A can be expressed as $A = \sigma_2\sigma_1$ with each σ_j of order two in two different ways. First, we choose two points w_1 and w_2 on $\mathcal{L}_A$ with w_2 halfway between w_1 and Aw_1 (so $\rho(w_1, w_2) = \frac{1}{2}T_A$). Note that w_1 or w_2 can be chosen arbitrarily and the remaining w_j is then uniquely determined. The first factorisation of A is $A = \sigma_2\sigma_1$ where σ_j is a rotation of order two about w_j. The second factorisation is obtained by constructing the geodesics L_j through w_j and orthogonal to $\mathcal{L}_A$. Then $A = \sigma_2\sigma_1$ where σ_j now denotes reflection in L_j.

Received by the editors July 23, 1981.
1980 *Mathematics Subject Classification*. Primary 53A35.

The essential step in [1] is to prove that if A, B and ABA are hyperbolic with axes $\mathcal{L}_A$, $\mathcal{L}_B$ meeting at w, then $\mathcal{L}_{ABA}$ also passes through w. In fact, ABA is necessarily hyperbolic (whether or not $\langle A, B\rangle$ is discrete) and we can give a simpler proof of a more general result.

THEOREM 1. *Let A and B be isometries with no common fixed point and let $C = ABA$.*

(i) *If A and B are hyperbolic with $\mathcal{L}_A$ and $\mathcal{L}_B$ meeting at w in Δ, then C is hyperbolic and $\mathcal{L}_C$ also passes through w.*

(ii) *If A and B are hyperbolic with $\mathcal{L}_A$ and $\mathcal{L}_B$ disjoint, let L be the unique geodesic orthogonal to $\mathcal{L}_A$ and $\mathcal{L}_B$. If C is hyperbolic then $\mathcal{L}_C$ is also orthogonal to L; if C is elliptic or parabolic then the fixed point of C lies on L.*

(iii) *If A and B are elliptic, let L be the geodesic containing the fixed points of A and B. If C is hyperbolic then $\mathcal{L}_C$ is orthogonal to L; if C is elliptic or parabolic then the fixed point of C lies on L.*

(iv) *If one of A and B is hyperbolic and the other is elliptic or parabolic, let L be the geodesic through the elliptic or parabolic fixed point and orthogonal to the hyperbolic axis. If C is hyperbolic then $\mathcal{L}_C$ is orthogonal to L; if C is elliptic or parabolic then the fixed point of C lies on L.*

PROOF. If A and B are hyperbolic then, as A and B have no common fixed point, their axes either meet in Δ (as in (i)) or have a common orthogonal (as in (ii)).

We show that in every case we can write

$$(1) \qquad A = \sigma_1\sigma_2, \qquad B = \sigma_2\sigma_3$$

where σ_1, σ_2 and σ_3 are either all rotations of order two or all reflections in geodesics. Then

$$ABA = (\sigma_1\sigma_3\sigma_1)\sigma_2 = \sigma_4\sigma_2$$

say. If the σ_j $(j = 1,2,3)$ are rotations of order two then so is σ_4. Note that in this case, if w_j is the fixed point of σ_j, then $w_4 \neq w_2$ (because $w_4 = \sigma_1(w_3)$ and so $w_4 = w_2$ implies that w_1, w_2, w_3 are collinear which, in turn, implies that $\mathcal{L}_A = \mathcal{L}_B$). It follows that in this case, ABA is hyperbolic with $\mathcal{L}_{ABA}$ passing through the fixed point of σ_2.

If the σ_j $(j = 1,2,3)$ are reflections, then so is σ_4, and ABA is elliptic or parabolic when L_4 and L_2 meet (L_j being the line of fixed points of σ_j), and it is hyperbolic when L_4 and L_2 are disjoint. In the first case, the fixed point of ABA is the point $L_4 \cap L_2$ and this is on L_2; in the second case $\mathcal{L}_{ABA}$ is orthogonal to L_2 (and to L_4).

These remarks combined with the following explicit description of the factorisation (1) in each of the cases (i)–(iv) yield the desired result. In case (i), we simply put $w_2 = w$ and let σ_2 be the rotation of order two about w_2. The rotations σ_1 and σ_3 are chosen appropriately with fixed points on $\mathcal{L}_A$ and $\mathcal{L}_B$ respectively and (as already noted) $\mathcal{L}_{ABA}$ passes through w. In cases (ii), (iii) and (iv) we simply let σ_2 be the reflection in L and the proof is complete.

3. Lie products. We have deliberately proved Theorem 1 without reference to Lie products; we now relate the proof of Theorem 1 to the concept of a Lie product. First, we briefly review the geometry of hyperbolic space.

Every Möbius transformation

$$A: z \mapsto \frac{az+b}{cz+d}, \qquad ad - bc = 1,$$

extends uniquely to act as a Möbius transformation on

$$H^3 = \{(z, t): z \in C, t > 0\}$$

and the action is given by identifying (z, t) with the quaternion

$$\zeta = x + iy + tj + Ok$$

and writing $A: \zeta \mapsto (a\zeta + b)(c\zeta + d)^{-1}$, the computation being within the algebra of quaternions.

If A and B leave a disc Δ invariant, we can (by conjugation) assume that A and B are in SL(2, R) and that Δ is $\{x + iy: y > 0\}$. It is equally acceptable (and for our purposes, preferable) to consider A and B as isometries of the model

$$\Delta' = \{x + tj: x \in R^1, t > 0\}$$

(a vertical section of H^3) for A and B act on Δ' in the same way as they act on Δ.

Now any hyperbolic or elliptic X in SL(2, C) has an axis $\mathcal{H}_X$ in H^3 which is the geodesic in H^3 joining the fixed points of X. If X is hyperbolic and in SL(2, R) then the axes $\mathcal{L}_X$ in Δ' and $\mathcal{H}_X$ in H^3 are the same geodesic. If X is elliptic and in SL(2, R) then X has a unique fixed point w in Δ' and $\mathcal{H}_X$ is the geodesic in H^3 orthogonal to Δ' and passing through w.

Given matrices A and B in SL(2, C), their Lie product is

$$\varphi(A, B) = AB - BA.$$

We can view this matrix as a Möbius transformation provided that it is nonsingular and this is so precisely when A and B have no common fixed point (which we shall assume). As trace $\varphi(A, B) = 0$, we see that $\varphi(A, B)$ is elliptic of order two. Moreover (see [1]) $\varphi(A, B) = \varphi(B^{-1}, A) = -\varphi(B, A)$ and so

$$A^{-1}\varphi(A, B)A^{-1} = A^{-1}\varphi(B^{-1}, A)A^{-1} = \varphi(A^{-1}, B^{-1}) = \varphi(A, B).$$

Thus with $\varphi = \varphi(A, B)$, we have $\varphi A^{-1}\varphi^{-1} = A$ and (similarly) $\varphi B^{-1}\varphi^{-1} = B$. This shows that if A and B are hyperbolic or elliptic, then φ interchanges the two fixed points of A and of B and so the axis of φ in H^3 is orthogonal to the axes $\mathcal{H}_A$ and $\mathcal{H}_B$. In fact, this orthogonal is unique so $\varphi(A, B)$ is actually determined by $\mathcal{H}_A$ and $\mathcal{H}_B$ alone.

Now assume that A and B are in SL(2, R) and have no common fixed points. If A and B are hyperbolic with $\mathcal{L}_A$ and $\mathcal{L}_B$ meeting at w, then their Lie product restricted to Δ' is a rotation of Δ' of order two about w. If $\mathcal{L}_A$ and $\mathcal{L}_B$ are disjoint, they have a common orthogonal L in Δ' and $\varphi(A, B)$ is a rotation of H^3 of order two with axis L. In this case, the restriction of $\varphi(A, B)$ to Δ' is precisely the reflection in L.

If A and B are elliptic there is a unique geodesic L in Δ' containing the fixed points of A and B in Δ' and this is the axis of $\varphi(A, B)$. Again $\varphi(A, B)$ leaves Δ' invariant and acts on Δ' as a reflection in L.

Finally, suppose that one of A and B is hyperbolic with axis $\mathcal{L}$ and that the other is elliptic with fixed point w in Δ'. Let L be the geodesic in Δ' through w and orthogonal to L: then L is the axis of $\varphi(A, B)$ and $\varphi(A, B)$ acts on Δ' as a reflection in L. These observations constitute a proof of the following result.

THEOREM 2. *Let A and B be in* SL(2, R) *and be either hyperbolic or elliptic. Then the Lie product $\varphi(A, B)$ leaves Δ' invariant and its action on Δ' is precisely that of σ_2 described in the proof of Theorem* 1.

In fact, it is possible (though probably not useful) to represent all of the σ_j described earlier in terms of Lie products. Suppose for example, that A and B are hyperbolic with $\mathfrak{L}_A$ and $\mathfrak{L}_B$ meeting at w. Then $A = \sigma_1\sigma_2$ and A has a square root $A^{1/2}$ (with the same axis but one half of the translation length of A) and $X = A^{1/2}B(A^{1/2})^{-1}$ is hyperbolic with its axis meeting $\mathfrak{L}_A$ at w_1. We deduce that $\sigma_1 = \varphi(A, X)$.

4. Matrix computation. It is possible to prove Theorem 1 simply by computing matrices. For example, to prove (i) we may assume that Δ is the unit disc and that $w = 0$. It is then only necessary to show that the sum of the fixed points of ABA is zero and this is easily established from the coefficients of ABA.

5. Geodesics on a Riemann surface. Let R be any Riemann surface with the hyperbolic plane as its universal covering surface. Exactly as in [**1**] we observe that if two closed geodesics on R intersect at w, say, then infinitely many closed geodesics on R pass through w. This is Theorem 1(i): note that the remark [**1**, p. 141] is a direct consequence of our factorisation (1). Theorem 1(ii) also yields a result about geodesics on R, namely given two closed disjoint geodesics L_1 and L_2 on R then there are infinitely many closed geodesics orthogonal to the unique geodesic L which is itself orthogonal to L_1 and L_2. The equivalent statement to the remark [**1**, p. 141] is that if, in Theorem 1(ii), the isometries A and B are oriented in opposite directions, then $\mathfrak{L}_{ABA}$ lies between $\mathfrak{L}_A$ and $\mathfrak{L}_B$. Note also that we could take B to be a conjugate of A; thus the above interpretation also holds when $L_1 = L_2$. Clearly Theorem 1(iii) and (iv) also have similar interpretations in terms of quotient spaces by Fuchsian groups with elliptic elements.

REFERENCES

1. T. Jørgensen, *Closed geodesics on Riemann surfaces*, Proc. Amer. Math. Soc. **72** (1978), 140–142.

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, 16 MILL LANE, CAMBRIDGE CB2 1SB, ENGLAND