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MAXIMAL EXTENSIONS OF FIRST-COUNTABLE SPACES

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ABSTRACT. A first-countable space is called maximal if it is not contained as a dense subspace in a first-countable space properly. The following are shown; (1) every locally compact, first-countable space is a dense subspace of a maximal space, (2) every metrizable space is a dense subspace of a maximal space, and (3) there is a first-countable space which is not a dense subspace of any maximal space.

All spaces in this paper are Tychonoff unless otherwise specified.

A space which contains a space X as a dense subspace is called an extension of X. Let us call a first-countable space *maximal*, or, more precisely, *maximal with respect to first-countability*, if it has no proper, first-countable extension. According to [S, Theorem 2.9], a first-countable space is maximal if and only if it is pseudocompact. (Note that our maximal spaces are identical to Stephenson's first countable- and completely regular-closed spaces.) Hence, if a first-countable space X has first-countable compactification Y, then Y is a maximal extension of X. On the other hand, even if X does not have a first-countable compactification, it can still have a maximal extension.

Here, we are concerned with two questions. Namely, which first-countable spaces have maximal extensions, and whether all first-countable spaces have maximal extensions.

In §§1 and 2, we shall answer the first question by showing that every locally compact, first-countable space and every metrizable space have maximal extensions. §3 will provide a negative answer to the second question.

1. Locally compact, first-countable spaces.

1.1. We shall begin with

PROPOSITION. *For a first-countable space X, the following conditions are equivalent*:

(1) *X has a maximal extension*,

(2) *there are a maximal disjoint collection $\mathfrak{Z} = \{Z_\alpha \mid \alpha \in A\}$ of zero-sets of βX and a family $\mathfrak{F} = \{f_\alpha \mid \alpha \in A\}$ of real-valued continuous functions on βX such that*

(a) *$\{x\} \in \mathfrak{Z}$ for each $x \in X$*,

(b) *$f_\alpha^{-1}(0) = Z_\alpha$ for each $\alpha \in A$, and*

(c) *$f_\alpha \mid Z_\beta$ is constant for any $\alpha \in A$ and $\beta \in A$.*

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PROOF. (1) → (2) Let Y be a maximal extension of X. Then Y is first-countable and pseudocompact. Since βY is a compactification of X, we have a continuous surjection h: $\beta X \to \beta Y$ which is identical on X. For each $y \in Y$, there is a real-valued continuous function f_y on βY such that $f_y^{-1}(0) = \{y\}$. Let $\mathfrak{Z} = \{h^{-1}(y) \mid y \in Y\}$ and $\mathfrak{F} = \{f_y \circ h \mid y \in Y\}$. Then $\mathfrak{Z}$ and $\mathfrak{F}$ satisfy the conditions (a) to (c). To see that $\mathfrak{Z}$ is a maximal disjoint collection of zero-sets of βX it suffices to observe that $h^{-1}(Y)$ is pseudocompact. This is shown by noting that the restriction $h \mid h^{-1}(Y)$ is perfect and irreducible, and that the preimage of a pseudocompact space under a perfect and irreducible map is always pseudocompact.

(2) → (1) Let $W = \bigcup\{Z_\alpha \mid \alpha \in A\}$ and Y be a quotient space of W obtained by collapsing each Z_α to a point. Obviously, Y is a first-countable, Tychonoff extension of X. The maximality of $\mathfrak{Z}$ implies that W is pseudocompact, hence so is Y. Therefore Y is a maximal extension of X.

1.2. THEOREM. *Every locally compact, first-countable space has a maximal extension.*

PROOF. Let X be locally compact and first-countable. At first consider a cozero-set C of $X^* = \beta X - X$ and a zero-set Z of βX such that $C \subset Z \subset X^*$. Then there is a continuous function g: $X^* \to \mathbf{R}$ such that $C = g^{-1}(\mathbf{R} - \{0\})$. (Here $\mathbf{R}$ denotes the space of reals.) Since X is locally compact, X^* is a closed subset of a compact space βX and g has a continuous extension h: $\beta X \to \mathbf{R}$. Take any $r \neq 0$. We have that $g^{-1}(r)$ is a G_δ-set of X^*, that Z is a G_δ-set of βX and that $g^{-1}(r) \subset Z \subset X^*$. Hence $g^{-1}(r)$ is a G_δ-set of βX and is expressed as $g^{-1}(r) = \bigcap\{U_n \mid n = 1,2,\dots\}$ with open sets U_n of βX. For each n, $h^{-1}(r) - U_n$ and X^* are disjoint closed subsets of βX. There is a continuous map k_n: $\beta X \to [0,1]$ such that $k_n(X^*) = \{0\}$ and $k_n(h^{-1}(r) - U_n) = \{1\}$. Define $f = \sum_{n=1}^{\infty}(1/2^n)k_n + |h - r|$. Then f is a real-valued continuous function on βX such that $f \mid X^* = |g - r|$ and $f^{-1}(0) = g^{-1}(r)$.

Next, consider the collection of all cozero-sets C of X^* such that $C \subset Z \subset X^*$ for some zero-set Z of βX. Take a maximal disjoint subcollection $\{C_\lambda \mid \lambda \in \Lambda\}$ of this collection. For each $\lambda \in \Lambda$, there is a continuous function g_λ: $X^* \to \mathbf{R}$ such that $C_\lambda = g_\lambda^{-1}(\mathbf{R} - \{0\})$. As above, for each $\lambda \in \Lambda$ and $r \neq 0$, there is a continuous function $f_{\lambda,r}$: $\beta X \to \mathbf{R}$ such that $f_{\lambda,r} \mid X^* = |g_\lambda - r|$ and $f_{\lambda,r}^{-1}(0) = g_\lambda^{-1}(r)$. Hence $Z_{\lambda,r} = g_\lambda^{-1}(r)$ is a zero-set of βX and $f_{\lambda,r} \mid Z_{\lambda',r'}$ is constant for any (λ, r) and (λ', r').

Moreover, if $Z \subset X^*$ is a zero-set of βX, then $\operatorname{int}_{X^*} Z \neq \varnothing$ because X is locally compact (see the proof of [W, 4.21]). Therefore $\operatorname{int}_{X^*} Z$, and hence Z, contains a cozero-set of X^*. This means that $\{\{x\} \mid x \in X\} \cup \{Z_{\lambda,r} \mid (\lambda, r) \in \Lambda \times (\mathbf{R} - \{0\})\}$ forms a maximal disjoint collection of zero-sets of βX. For each $x \in X$ let f_x be a real-valued continuous function on βX such that $f_x^{-1}(0) = \{x\}$ and $f_x(X^*) = \{1\}$. Then it is now easy to check that

$$\{\{x\} \mid x \in X\} \cup \{Z_{\lambda,r} \mid (\lambda, r) \in \Lambda \times (\mathbf{R} - \{0\})\}$$

and

$$\{\{f_x\} \mid x \in X\} \cup \{f_{\lambda,r} \mid (\lambda, r) \in \Lambda \times (\mathbf{R} - \{0\})\}$$

satisfy condition (2) of Proposition 1.1.

1.3. A space is called locally pseudocompact if every point has a pseudocompact neighborhood. For any locally pseudocompact space X we can find a locally compact space Y such that $X \subset Y \subset \upsilon X$. In 1.2, let us consider cozero-sets of $\beta X - Y$ instead of X^* and note that every zero-set of βX which meets $Y \subset \upsilon X$ meets X. Then we get an extension of X as a quotient space of a subspace of $X \cup (\beta X - Y)$. That is,

THEOREM. *Every locally pseudocompact, first-countable space has a maximal extension.*

1.4. Since $f_{\lambda,r}^{-1}([-r/2, r/2]) \subset X \cup C_\lambda$, the maximal extension of X in 1.2 is indeed locally compact. That is,

THEOREM. *Every locally compact, first-countable space has a locally compact, maximal extension.*

2. Metrizable spaces.

2.1. Let X be a space and A its dense subspace. If every sequence in A contains a subsequence which converges in X, then we call X *e-sequentially compact with respect to A*. It is not difficult to observe that

(1) if X is e-sequentially compact with respect to some dense subset, then X is pseudocompact,

(2) if X is e-sequentially compact with respect to A, and $B \subset A$, then $\mathrm{cl}_X B$ is e-sequentially compact with respect to B, and

(3) if X_n is e-sequentially compact with respect to A_n for each $n = 1, 2, \ldots$, then so is the product $\prod_{n=1}^{\infty} X_n$ with respect to $\prod_{n=1}^{\infty} A_n$.

2.2. Let us recall the following. Let A be a set of cardinality $\mathfrak{m} \geq \aleph_0$. The star-space $S(A)$ is the metric space $\{0\} \cup \bigcup\{(0,1] \times \{\alpha\} \mid \alpha \in A\}$ in which the metric function d is defined by: $d(0,(t,\alpha)) = t$, $d((t,\alpha),(s,\beta)) = |t - s|$ if $\alpha = \beta$, and $= t + s$ if $\alpha \neq \beta$. In [**E**, Example 4.1.5], the star-space is called the hedgehog space of spininess $\mathfrak{m}$. According to [**N**, Theorem VI.9], any metrizable space of weight $\leq \mathfrak{m}$ is homeomorphic to a subspace of the product of countably many copies of $S(A)$.

2.3. We shall show

PROPOSITION. *For each A, there is a first-countable space X which is e-sequentially compact with respect to $S(A)$.*

2.4. By virtue of 2.1 and 2.2, Proposition 2.3 immediately implies

THEOREM. *Every metrizable space M has a maximal extension which is e-sequentially compact with respect to M.*

2.5. PROOF OF PROPOSITION 2.3. Consider A as a discrete space based on the set A. Then A can be embedded as a dense subspace in a first-countable pseudocompact space B. (This can be shown in many ways. For example, apply Theorem 1.2 in §1, or see the proof of [**A**, 2.3.16], or take a maximal almost-disjoint collection $\mathfrak{R}$ of countably infinite subsets of A and topologize $A \cup \mathfrak{R}$ just as the spaces $N \cup \mathfrak{R}$ in [**T**].) Take the set $X = \{0\} \cup \bigcup\{(0,1] \times \{\beta\} \mid \beta \in B\}$. We topologize X in the following way: The point 0 has the neighborhood basis

$$\{0\} \cup \bigcup \{(0, 1/n) \times \{\beta\} \mid \beta \in B\}, \qquad n = 1, 2, \ldots,$$

and $(t, \beta) \in (0, 1] \times \{\beta\}$ has the neighborhood basis

$$((t - 1/n, t + 1/n) \cap (0, 1]) \times U_n, \qquad n = 1, 2, \ldots,$$

where $\{U_n \mid n = 1, 2, \ldots\}$ is a neighborhood basis at the point β in B. Now, it is easy to check that X is a first-countable Tychonoff space and is e-sequentially compact with respect to $S(A)$.

3. First-countable spaces without maximal extensions.

3.1. In **[vDP]**, a first-countable Lindelöf space Δ is constructed so that all compactifications of Δ contain βN. This Δ has no maximal extension. The authors incidentally, do not know if there exists a maximal space every compactification of which contains βN.

3.2. Δ is not Čech-complete. In view of Theorem 1.2 in §1, we present a Čech-complete modification Δ' of Δ. Let N be the set of natural numbers and take any almost-disjoint collection $\mathfrak{R}$ of infinite subsets of N such that $|\mathfrak{R}| = c$. (A collection of sets is called almost-disjoint if the intersection of any two of its elements is a finite set.) The collection of all subsets of N has the cardinality c and can be denoted by $\{A_\lambda \mid \lambda \in \mathfrak{R}\}$. Topologize the set $\Delta' = \mathfrak{R} \cup N \times N$ in the following way. All points of $N \times N$ are isolated and the neighborhood basis at $\lambda \in \mathfrak{R}$ consists of sets $\{\lambda\} \cup ((\lambda - (\text{finite set})) \times A_\lambda)$. It can be easily seen that Δ' is 0-dimensional, Tychonoff and first-countable. To see that Δ' is Čech-complete, let

$$\mathfrak{U}_n = \{\{\lambda\} \cup ((\lambda - \{1, 2, \ldots, n\}) \times A_\lambda) \mid \lambda \in \mathfrak{R}\}$$

and

$$\mathfrak{V}_n = \mathfrak{U}_n \cup \{\{x\} \mid x \in \Delta' - \bigcup \mathfrak{U}_n\} \quad \text{for } n = 1, 2, \ldots.$$

Then each $\mathfrak{V}_n$ is an open cover of Δ' and $\{\mathfrak{V}_n \mid n = 1, 2, \ldots\}$ determines Čech-completeness of Δ' by **[E, Theorem 3.9.2]**. Finally, let us show that Δ' has no first-countable, pseudocompact regular extension. We take any first-countable pseudocompact extension X of Δ' and prove that X is not regular. First, decompose N into countably many, disjoint, infinite sets: $N = \bigcup_{n=1}^{\infty} B_n$. Each $\{n\} \times N$ is a clopen (i.e., simultaneously-closed-and-open) set of Δ', hence so is $\{n\} \times B_n$. Since X is pseudocompact, $\{n\} \times B_n$ cannot be a closed set of X. Pick an accumulation point x_n of $\{n\} \times B_n$ in X for each n. It follows from the first-countability of X that there is an infinite subset $C_n \subset B_n$ such that $\{n\} \times C_n$, as a sequence, converges to x_n. Decompose C_n into two disjoint, infinite sets: $C_n = D_n \cup D_n'$. There is a $\lambda \in \mathfrak{R}$ such that $A_\lambda = \bigcup_{n=1}^{\infty} D_n$. Obviously $A_\lambda \cap \bigcup_{n=1}^{\infty} D_n' = \varnothing$. Now assume that X is regular. Then open sets U_n, $n = 1, 2, \ldots$, of X, each of which satisfies $\{\lambda\} \cup ((\lambda - \{1, 2, \ldots, n\}) \times A_\lambda) = U_n \cap \Delta'$, form a neighborhood basis at λ in X. Again, by the regularity of X, $\mathrm{cl}_X U_n \subset U_1$ for some n. Take any $m \in \lambda - \{1, 2, \ldots, n\}$. Then $\{m\} \times D_m \subset U_n$. On the other hand, $X - U_1 \supset \Delta' - U_n \supset \{m\} \times D_m'$. Here is a contradiction because both $\{m\} \times D_m$ and $\{m\} \times D_m'$ converge to $x_m \in X$.

3.3. It is also seen that all compactifications of Δ' contain βN. In order to obtain a first-countable, Lindelöf, Čech-complete space X without maximal extension, we employ Alexandroff-Urysohn's two arrow space [**E**, Exercise 3.10.C]. That is, define

$$X = (0,1] \times \{0\} \cup [0,1) \times \{1\} \cup Q \times N,$$

where Q denotes the set of rational numbers in $[0,1]$. All points of $Q \times N$ are defined to be isolated. To define neighborhoods at other points we need some notation. For $t \in [0,1]$, let $\lambda^+(t)$ and $\lambda^-(t)$ be strictly increasing and decreasing sequences, respectively, of rational numbers which converge to t. We assume that $\lambda^+(0) = \lambda^-(1) = \varnothing$. Let $\{A(x) \mid x \in (0,1] \times \{0\} \cup [0,1) \times \{1\}\}$ be the collection of all subsets of N. The neighborhood basis at $(t,0) \in (0,1] \times \{0\}$ is defined as U_n, $n = 1,2,\dots$, where U_n is the union of $\{(t,0)\}$, $\{(s,i) \mid t - 1/n < s < t \text{ and } i = 0,1\}$, $(\lambda^-(t) - (\text{finite set})) \times A(t,0)$ and $\bigcup\{\{q\} \times N \mid t - 1/n < q < t\} - \lambda^+(t) \times A(t,1)$. Neighborhood bases at points of $[0,1) \times \{1\}$ are defined likewise. The Čech-completeness and nonexistence of a maximal extension of X could be shown in the same way as 3.2.

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