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A PLÜCKER EQUATION FOR CURVES IN REAL PROJECTIVE SPACE

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ABSTRACT. For smooth closed curves in real projective space we write an equation relating the Plücker characteristics, various winding numbers, and a new characteristic involving pairs of points on the line at infinity. This theorem is a generalization of the Umlaufsatz and also relates directly to Plücker's equations for algebraic curves.

Introduction. For regular closed curves in the plane there is an equation relating the number of cusps and double points counted with a certain multiplicity (Plücker characteristics), the winding number about the initial point, and the tangent winding number. Various versions of this can be found in Hopf [**3**], Whitney [**10**], and Titus [**9**].

For curves in projective space, we define a new characteristic counting pairs of points on the line at infinity. Also we define the other characteristics in the context of projective space and its basic operations of duality and wedge product. Our theorem has the advantage that it may be applied to the dual curve to yield further information about the original curve. Also it links directly with Plücker's classical equations for algebraic curves (Griffiths and Harris [**2**]). The exact way that this method of proof can be adapted to prove Plücker's equations is discussed in Quine [**6**]. Also there we discuss generalizations of Plücker's equations to value distribution theory in complex analysis (Cowen and Griffiths [**1**]; Quine [**5**]; Quine and Yang [**7**]).

Notation. Let P^2 be the real projective plane and S^2 be the unit sphere in R^3. We have the quotient maps $R^3 - \{0\} \to S^2 \to P^2$. Let $a = (a_1, a_2, a_3)$ be a point in $R^3 - \{0\}$. We may think of this as homogenous coordinates of a point in S^2 or P^2. For simplicity of notation, we will write a for the point in $R^3 - \{0\}$, or its image in either S^2 or P^2, depending on the context. A point in S^2 will sometimes be referred to as an oriented point in P^2. (Since P^2 is not orientable, no orientation can be chosen continuously for all points in P^2, but an individual point may be given one or the other orientation.) Let e_1, e_2 and e_3 be the standard basis in R^3. Let $a \cdot b$ be the standard dot product in R^3. The dot product can be said to be only positive, negative, or zero in S^2, and zero or nonzero in P^2. Thus for a and b in $R^3 - \{a\}$ we may meaningfully write $\operatorname{sgn} a \cdot b$.

By the principle of duality, lines in P^2 and points in P^2 are in one-to-one correspondence. Corresponding to the point a, is the line $a^\perp = \{b \mid b \cdot a = 0\}$, which

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may also be thought of as a plane in R^3. Let $a \wedge b$ be the usual cross product in R^3. Then if a and b are distinct in P^2, $(a \wedge b)^\perp$ is the line joining a and b. Also $a \wedge b$ is the point of intersection of $a^\perp$ and $b^\perp$. We call $\{b^\perp | b \cdot a = 0\}$ the pencil of lines through a.

It will be necessary to understand the relationship between oriented points and oriented lines. If a is in $R^3 - \{0\}$, then a determines an orientation on the plane $a^\perp$ by the right-hand rule, i.e., if the ordered basis (a, b, c) is positively oriented in R^3, with b and c in $a^\perp$, then the ordered basis (b, c) will determine positive orientation in $a^\perp$. Now counterclockwise orientation in the plane $a^\perp$ determines an orientation on $a^\perp$ in P^2 and S^2, so that $a^\perp$ may be thought of as an oriented copy of S^1, or an oriented line. Thus we have a one-to-one correspondence between oriented points and oriented lines, and an oriented point may be thought of as a point in P^2 together with an orientation on $a^\perp$ (or an orientation on the pencil of lines through a). If a and b are oriented points, distinct in P^2, we have an orientation given on $(a \wedge b)^\perp$ by the oriented basis (a, b). This corresponds to the orientation given to $a \wedge b$ by the right-hand rule.

We will think of R^2 as a subset of P^2 by identifying the point (x, y) in R^2 with the point $(x, y, 1)$ in R^3. Thus e_3 corresponds to the origin in R^2. P^2 an be thought of as R^2 and the line at infinity, $e_3^\perp$.

Main theorem. We will consider a smooth curve in P^2 considered as a smooth map f of period 1 from the reals R into P^2. Equivalently, we have a smooth map from the circle S^1 into P^2. The point $f(0)$ is called the initial point. Locally, f is given by a map from an interval into $R^3 - \{0\}$. For convenience of notation we also call these local maps f, and we may speak of the derivative f'. We will call $f(p)$ a regular point if $f'(p)$ and $f(p)$ are linearly independent, and a cusp point if they are linearly dependent but $f(p)$ and $f''(p)$ are linearly independent. It is easily verified that these conditions are independent of the local representation for f, in the sense that if f is replaced by hf where h is a smooth real valued function which is nowhere zero, the conditions hold for f if and only if they hold for hf. We will assume that f has only regular or cusp points. We say that $f(p) = f(q)$ is a simple double point if $f'(p)$ and $f'(q)$ are linearly independent and $f(p)$ has only two preimage points in the interval $(0, 1]$. We will assume that the only points with more than one preimage point are simple double points.

Corresponding to f, we define a dual curve $\hat{f}$ in P^2 as follows: If $f(p)$ is not a cusp, $\hat{f}$ is defined by $f(q) \wedge f'(q)$ for q near p. If $f(p)$ is a cusp, $\hat{f}$ is defined by $f(q) \wedge f'(q)/(q-p)$ for q near p. From the latter, it follows that $\hat{f}(p) = f(p) \wedge f''(p)$. We see that $(\hat{f}(p))^\perp$ is the tangent line to f at $f(p)$ and that by this process we have defined a tangent line at each cusp point.

If $f(p)$ is not a cusp, $f(p) \wedge f'(p)$ defines a point in $\tilde{P}^2$ which is independent of the choice of the local representation $f(p)$. This is because replacing f by hf replaces the point by $(h(p))^2 f(p) \wedge f'(p)$. Thus if $f(p)$ is not a cusp, we may think of $\hat{f}(p)$ as a point in S^2 giving a directed tangent line corresponding to the direction of the curve. Although $\hat{f}$ is a smooth curve in P^2, it is not necessarily a smooth curve in S^2 since the tangent line changes direction at cusps.

We can also look at the dual of the dual, $\hat{\hat{f}}$. Using the cross product formula

$$(a \wedge b) \wedge (c \wedge d) = (a \wedge b \cdot d)c - (a \wedge b \cdot c)d \tag{1}$$

we see that in the neighborhood of a regular point $\hat{\hat{f}}(p) = w(p)f(p)$ where $w(p) = (f(p) \wedge f'(p) \cdot f''(p))$ is the Wronskian. If $f(p)$ is a cusp the local representation becomes $(w(q)/(q-p)^2)f(q)$ for q near p, and

$$\hat{\hat{f}}(p) = (f(p) \wedge f''(p) \cdot f'''(p))f(p).$$

Thus $\hat{\hat{f}}$ is the same as f in P^2. As a dual curve, however, $\hat{\hat{f}}$ may be thought of as lying in $\tilde{P}^2$ except at cusps of $\hat{f}$ (inflection points of f). The oriented point $\hat{\hat{f}}(p)$ indicates the direction of the curvature of f by indicating an orientation on the pencil of lines through $f(p)$.

We now define the winding number of f about an oriented point a, denoted $n(a)$. We consider the map $f \wedge a$: $S^1 \to a^\perp$ and think of $a^\perp$ as an oriented copy of S^1. The integer $n(a)$ is defined to be the degree of this map. We see that this definition makes sense even if a is on the curve. For curves in R^2 with the point a not on the curve and $a \cdot e_3 > 0$, $\frac{1}{2}n(a)$ is the usual winding number. The winding number of the dual curve around an oriented point a is denoted $\hat{n}(a)$. We note that $\frac{1}{2}\hat{n}(e_3)$ is what is sometimes called the tangent turning number, or tangent winding number.

The conclusion of the theorem is based on an index of ± 2 we assign to each double point and ordered pairs of points on the line at infinity, and an index of ± 1 that we assign to each cusp. We assume that no double points or cusps are on the line at infinity.

For each double point $f(p) = f(q)$ with $0 < p < q \leqslant 1$ we set

$$\delta_{p,q} = -2 \operatorname{sgn} \hat{f}(p) \wedge \hat{f}(q) \cdot e_3.$$

This indicates the orientation of the pencil $f(p)^\perp = f(q)^\perp$ as determined by the oriented tangent lines $\hat{f}(p)^\perp$ and $\hat{f}(q)^\perp$ at that point. To assign an index to each pair of points on the line at infinity, we make the additional assumption that e_3 is not on the curve $\hat{f}$ in P^2. This essentially means that f meets the line at infinity transversely. In this case, for every pair of points $f(p)$ and $f(q)$ with $0 < p < q \leqslant 1, f(p) \cdot e_3 = 0$ and $f(g) \cdot e_3 = 0$, we set

$$\nu_{p,q} = 2 \operatorname{sgn} \hat{f}(p) \wedge \hat{f}(q) \cdot e_3.$$

Again we are looking at the orientation determined by $\hat{f}(p)^\perp$ and $\hat{f}(q)^\perp$ on the intersection point of these two tangent lines. We next discuss the index at each cusp point. The definition of a different sort, and the reason for this will appear in the proof. We must modify f at the cusp point by rounding off the curve. In other words, if $f(p)$ is a cusp, $0 < p \leqslant 1$, we remove the image of a small neighborhood of p and replace it by a simple arc in such a way that the resulting curve is still smooth. This will change $\hat{n}(e_3)$ by subtraction of an amount ± 1, and this amount we denote by k_p. If $\hat{f}(p)$ is not a cusp of $\hat{f}$, then k_p can be determined by the limiting direction of curvature at the cusp point, i.e., $k_p = 2 \operatorname{sgn} \hat{\hat{f}}(p) \cdot e_3$. Finally, we set $k = \sum k_p$, $\delta = \sum \delta_{p,q}$, and $\nu = \sum \nu_{p,q}$ where the sums are over the appropriate pairs (p, q) and points p. We remark that these integers depend on the parametrization of the curve, or equivalently on the choice of the initial point $f(0)$.

We are now ready to state

THEOREM. *Suppose f is a smooth curve in P^2 with only regular points or simple cusps and whose only multiple points are simple double points. Suppose the initial point in e_3 and that this is neither a double point nor a cusp. Suppose e_3 is not on the dual curve and no double points or cusps are on the line at infinity. Then*

$$k + \nu + \delta = \hat{n}(e_3) - 2n(e_3).$$

PROOF. We first prove the theorem for the case when f has no cusps. Let $T = \{(p, q) \mid 0 \leqslant p \leqslant q \leqslant 1\}$ and $D = \{(p, q) \in T \mid f(p) = f(q),\ p < q\}$. We define the secant map s from $T - D$ to P^2 by $s(p, q) = f(p) \wedge f(q)$ for $p \neq q$ and $s(p, q) = f(p) \wedge f(q)/(p - q)$ near the diagonal. Thus $s(p, p) = \hat{f}(p)$ and the secant map restricted to the diagonal is essentially the dual curve. Next we follow the secant map by projection onto the line at infinity, i.e. we look at the map $s \wedge e_3$ defined on $T - (D \cup S)$ where $S = \{(p, q) \in T \mid f(p) \cdot e_3 = 0 = f(q) \cdot e_3, p < q\}$. Now the range of $s \wedge e_3$ is $e_3^{\perp}$ which we may consider as an oriented copy of S^1. The theorem is proved by comparing the degree of $s \wedge e_3$ on the boundary of T with the local degree at point in D and S. To investigate the local degree we look at the map $s \wedge e_3$ in a neighborhood of these points as a map into the oriented plane $e_3^{\perp}$ in R^3. At the point (p, q) in S or D, the map $s \wedge e_3$ is zero and if the map is nonsingular there, the local degree is just twice the sign of the Jacobian determinant. The factor two arises because the quotient map from $e_3^{\perp}$ in S^2 to $e_3^{\perp}$ in P^2 is a double covering. Now the map $s \wedge e_3$ is given locally in coordinates by $(f(p) \wedge f(q) \cdot e_2, -f(p) \wedge f(q) \cdot e_1)$ and the Jacobian determinant is

$$\begin{vmatrix} f'(p) \wedge f(q) \cdot e_2 & -f'(p) \wedge f(q) \cdot e_1 \\ f(p) \wedge f'(q) \cdot e_2 & -f(p) \wedge f'(q) \cdot e_1 \end{vmatrix} = (f'(p) \wedge f(q)) \wedge (f(p) \wedge f'(q)) \cdot e_3. \tag{2}$$

Now if $(p, q) \in D$ we may assume $f(p) = f(q)$ in S^2 and so (2) becomes $-\hat{f}(p) \wedge \hat{f}(q) \cdot e_3$ and the local degree is

$$-2 \operatorname{sgn} \hat{f}(p) \wedge \hat{f}(q) \cdot e_3 = \delta_{p,q}.$$

If $(p, q) \in S$, we use the cross product identity (1) to show that (2) is $\hat{f}(p) \wedge \hat{f}(q) \cdot e_3$ and so the local degree is

$$2 \operatorname{sgn} \hat{f}(p) \wedge \hat{f}(q) \cdot e_3 = \nu_{p,q}.$$

Now, restricted to the line $p = 0$, $s \wedge e_3$ becomes $(e_3 \wedge f(q)) \wedge e_3$ and the degree of this is $n(e_3)$. Likewise the degree of $s \wedge e_3$ restricted to the line $q = 1$ is $n(e_3)$. The degree of this map restricted to the diagonal is $\hat{n}(e_3)$. Now equating the sum of the local degrees of $s \wedge e_3$ at points of $S \cup D$ to the degree of the map restricted to the boundary of T, we get the result.

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