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PARACOMPACTNESS OF PIXLEY-ROY HYPERSPACES. I

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ABSTRACT. In this paper, we will study the relations between the paracompactness of a Pixley-Roy hyperspace $\mathcal{F}[X]$ and the paracompactness of the subspace $\mathcal{F}_2[X]$. We will give the affirmative answer to H. Bennett's problem.

1. Introduction. The *Pixley-Roy hyperspace* $\mathcal{F}[X]$ over a space X **[8]** is the set of all nonempty finite subsets of X with the topology generated by the sets of the form $[F, U] = \{G \in \mathcal{F}[X]\colon F \subset G \subset U\}$, where $F \in \mathcal{F}[X]$ and U is an open set in X containing F. Throughout this paper, all spaces are assumed to be T_1-spaces. In **[4]**, it was pointed out that $\mathcal{F}[X]$ is a zero-dimensional Tychonoff space. Let N denote the set of natural numbers. Notice that for each $n \in N$, $\mathcal{F}_n[X] = \{F \in \mathcal{F}[X]\colon \operatorname{card}(F) \leqslant n\}$ is a closed subspace of $\mathcal{F}[X]$ and, in particular, $\mathcal{F}_1[X]$ is a discrete closed subspace of $\mathcal{F}[X]$.

If X is a set linearly ordered by $<$, then X with the usual order topology $\lambda(<)$ induced by $<$ is called a *linearly ordered topological space* (= LOTS). Intervals are denoted in the usual way. For example, we denote $\{x \in X\colon a \leqslant x \leqslant b\}$ by $[a, b]$ for $a, b \in X$. A subset C of a LOTS X is called *order-convex* if whenever $a, b \in C$ have $a < b$, then $[a, b] \subset C$. If Y is a set linearly ordered by $<$ and τ is a topology on Y such that (1) $\lambda(<) \subset \tau$ and (2) τ has a base consisting of order-convex sets, then $X = (Y, \tau)$ is called a *generalized ordered space* (= GO space) **[6]**. We often say that the GO space X is contructed on the LOTS Y.

It has already been shown that some of the properties of $\mathcal{F}[X]$ can be studied in terms of the properties of the subspaces $\mathcal{F}_n[X]$. For example, $\mathcal{F}[X]$ is a Moore space if and only if $\mathcal{F}_2[X]$ is first countable if and only if X is first countable **[7]**. The following theorem was obtained in **[1]** or **[2]**.

THEOREM. *Let X be a GO space constructed on a separable LOTS. Then the following are equivalent.*

(a) $\mathcal{F}[X]$ *is metrizable, and*

(b) $\mathcal{F}_2[X]$ *is metrizable.*

In **[1]**, H. Bennett asked whether separability can be omitted from this theorem. The purpose of this paper is to give the affirmative answer to this problem.

If $\mathcal{W}$ is a collection of subsets of $\mathcal{F}[X]$, let $\mathcal{W}^* = \bigcup\{W\colon W \in \mathcal{W}\}$. For undefined terminology, the reader is referred to **[5]**.

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2. Relations between the paracompactness of $\mathfrak{F}[X]$ and the paracompactness of $\mathfrak{F}_2[X]$. After completing this paper, the author learned that T. Przymusiński proved Lemmas 1 and 2 by the same method. Interested readers should consult [**10**].

LEMMA 1. *Let $\mathcal{A}$ be an open-and-closed subset of $\mathfrak{F}[X]$ such that $\mathcal{A} \supset \mathfrak{F}_{n-1}[X]$, where $\mathfrak{F}_0[X] = \varnothing$. For each $F \in \mathfrak{F}_n[X] - \mathcal{A}$, let $U(F)$ be an open set in X containing F such that $[F, U(F)] \cap \mathcal{A} = \varnothing$. Let $\mathcal{W} = \{[F, U(F)]\colon F \in \mathfrak{F}_n[X] - A\}$. If $\mathcal{W}$ is a pairwise disjoint collection, then $\mathcal{W}$ is discrete.*

PROOF. It suffices to prove that $\mathcal{W}^*$ is closed in $\mathfrak{F}[X]$. Let $F \in \mathfrak{F}[X] - \mathcal{W}^*$. We can assume $F \notin \mathcal{A}$. Let $F = \{x_1, \ldots, x_m\}$, where $m > n$. If $N(F) = \{G \subset F\colon 0 < \operatorname{card}(G) \leq n\}$, then there exists a collection $\{B_F(x_i)\colon 1 \leq i \leq m\}$, where each $B_F(x_i)$ is an open set in X containing x_i, such that

(1) if $G \in N(F)$ and $G \in \mathcal{A}$, then $[G, \bigcup\{B_F(x)\colon x \in G\}] \subset \mathcal{A}$, and,

(2) if $G \in N(F)$ and $G \notin \mathcal{A}$, then $B_F(x) \subset U(G)$ for each $x \in G$.

Let $B(F) = \bigcup\{B_F(x_i)\colon 1 \leq i \leq m\}$ and let $\mathcal{O} = [F, B(F)]$. It follows that $\mathcal{O} \cap \mathcal{W}^* = \varnothing$. To see this, assume that there exists some $H \in \mathfrak{F}_n[X] - \mathcal{A}$ such that $\mathcal{O} \cap [H, U(H)] \neq \varnothing$. Then $H \subset B(F)$ and $F \subset U(H)$. For each $y \in H$, let x^y be an element of F such that $y \in B_F(x^y)$. If $G = \{x^y\colon y \in H\}$, then $\operatorname{card}(G) \leq n$. Hence $G \in N(F)$. Since

$$G \cup H \in \left[G, \bigcup \{B_F(x^y)\colon y \in H\}\right] \cap [H, U(H)],$$

it follows that $G \in \mathfrak{F}_n[X] - \mathcal{A}$. Thus, using (2), $[G, \bigcup\{B_F(x^y)\colon y \in H\}] \subset [G, U(G)]$. Hence $[G, U(G)] \cap [H, U(H)] \neq \varnothing$. Since $\mathcal{W}$ is a pairwise disjoint collection, it follows that $G = H$. Hence $H \subset F$. Since $F \subset U(H)$, $F \in [H, U(H)] \subset \mathcal{W}^*$. From this contradiction, it follows that $\mathcal{O} \cap \mathcal{W}^* = \varnothing$ and $\mathcal{W}^*$ is closed in $F[X]$.

A space X is called *collectionwise Hausdorff* if for each discrete closed subspace D of X, there exists a pairwise disjoint collection $\{U(d)\colon d \in D\}$ of open subsets of X such that $D \cap U(d) = \{d\}$ for each $d \in D$. Every paracompact Hausdorff space is collectionwise Hausdorff.

A space X is called *weakly separated* [**11**] if for each $x \in X$, there exists an open set $U(x)$ in X containing x such that if $y \in U(x)$ and $x \in U(y)$, then $x = y$.

LEMMA 2. *The following are equivalent.*

(a) *$\mathfrak{F}_3[X]$ is paracompact,*

(b) *$\mathfrak{F}_2[X]$ is paracompact,*

(c) *$\mathfrak{F}_2[X]$ is collectionwise Hausdorff, and*

(d) *X is weakly separated.*

PROOF. The implications (a) $\to$ (b) and (b) $\to$ (c) are obvious.

(c) $\to$ (d). Since $\mathfrak{F}_1[X]$ is a discrete closed subspace of $\mathfrak{F}_2[X]$, there exists a pairwise disjoint collection $\mathcal{V} = \{\mathcal{V}(x)\colon x \in X\}$ of open subsets of $\mathfrak{F}_2[X]$ such that $\mathcal{V}(x) \cap \mathfrak{F}_1[X] = \{\{x\}\}$ for each $x \in X$. Thus, for each $x \in X$, there exists an open set $U(x)$ containing x such that $[\{x\}, U(x)] \cap \mathfrak{F}_2[X] \subset \mathcal{V}(x)$. If $y \in U(x)$ and $x \in U(y)$, then

$$\{x, y\} \in [\{x\}, U(x)] \cap [\{y\}, U(y)] \cap \mathfrak{F}_2[X] \subset \mathcal{V}(x) \cap \mathcal{V}(y).$$

Since $\mathcal{V}$ is a pairwise disjoint collection, it follows that $\mathcal{V}(x) = \mathcal{V}(y)$ and $x = y$.

(d) → (a). Let $\mathfrak{U} = \{U(x)\colon x \in X\}$ be a collection of open subsets of X that weakly separates X. Let $\mathcal{V}$ be an open covering of $\mathcal{F}_3[X]$ and for each $x \in X$, let $\mathcal{V}(x)$ be an element of $\mathcal{V}$ such that $\{x\} \in \mathcal{V}(x)$. Let $W(x)$ be an open set in X containing x such that for each $x \in X$,

(1) $[\{x\}, W(x)] \cap \mathcal{F}_3[X] \subset \mathcal{V}(x)$, and

(2) $W(x) \subset U(x)$.

Let $\mathcal{W}_1 = \{[\{x\}, W(x)]\colon x \in X\}$. Clearly $\mathcal{W}_1$ is a pairwise disjoint collection in $\mathcal{F}[X]$ and hence, by Lemma 1, $\mathcal{W}_1^*$ is closed in $\mathcal{F}[X]$. Let $\mathcal{W}_1' = \{W \cap F_3[X]\colon W \in \mathcal{W}_1\}$. It follows that $\mathcal{W}_1'^*$ is closed in $\mathcal{F}_3[X]$ and hence, also closed in $\mathcal{F}[X]$. For each $F \in \mathcal{F}_2[X] - \mathcal{W}_1'^*$, let $\mathcal{V}(F)$ be an element of $\mathcal{V}$ such that $F \in \mathcal{V}(F)$. For each $F = \{x_1, x_2\} \in \mathcal{F}_2[X] - \mathcal{W}_1'^*$, let $W_F(x_1)$ and $W_F(x_2)$ be open subsets of X containing x_1 and x_2 respectively such that

(3) $W_F(x_1) \subset W(x_1)$ and $W_F(x_2) \subset W(x_2)$,

(4) $[F, W_F(x_1) \cup W_F(x_2)] \cap \mathcal{F}_3[X] \subset \mathcal{V}(F)$, and

(5) $[F, W_F(x_1) \cup W_F(x_2)] \cap \mathcal{W}_1'^* = \varnothing$.

For each $F = \{x_1, x_2\} \in \mathcal{F}_2[X] - \mathcal{W}_1'^*$, let $W(F) = W_F(x_1) \cup W_F(x_2)$. Let

$$\mathcal{W}_2 = \{[F, W(F)] \cap \mathcal{F}_3[X]\colon F \in \mathcal{F}_2[X] - \mathcal{W}_1'^*\}.$$

Suppose that $[F, W(F)] \cap [G, W(G)] \cap F_3[X] \neq \varnothing$ for some F and G in $\mathcal{F}_2[X] - \mathcal{W}_1'^*$. Then F and G contain at least one common element $z \in X$. If $F \neq G$, then $F = \{x, z\}$, $G = \{y, z\}$, and $x \neq y$. Since X is weakly separated, either $y \notin U(x)$ or $x \notin U(y)$. Assume that $y \notin U(x)$. Thus, by (2) and (3), $y \notin W_F(x)$. Since $G \in \mathcal{F}_2[X] - \mathcal{W}_1'^*$, $y \notin W(z)$. Thus, using (3), $y \notin W_F(z)$. Hence $y \notin W(F)$. From this contradiction, it follows that $F = G$. Thus $\mathcal{W}_2$ is a pairwise disjoint collection in $\mathcal{F}_3[X]$. Since each point of $\mathcal{F}_3[X] - (\mathcal{W}_1'^* \cup \mathcal{W}_2^*)$ is isolated in $\mathcal{F}_3[X]$, it follows that $\mathcal{W}_1' \cup \mathcal{W}_2 \cup \{\{F\}\colon F \in \mathcal{F}_3[X] - (\mathcal{W}_1'^* \cup \mathcal{W}_2^*)\}$ is a pairwise disjoint open refinement of $\mathcal{V}$. Hence $\mathcal{F}_3[X]$ is paracompact.

LEMMA 3. *Let $\mathcal{C}$ be an open-and-closed subset of $\mathcal{F}[X]$ such that $\mathcal{C} \supset \mathcal{F}_n[X]$. For each $F \in \mathcal{F}_{n+1}[X] - \mathcal{C}$, let $\{W_F(x)\colon x \in F\}$ be a collection of open subsets of X such that*

(1) *$x \in W_F(x)$ for each $x \in F$,*

(2) *$[F, \bigcup\{W_F(x)\colon x \in F\}] \cap \mathcal{C} = \varnothing$, and*

(3) *$[G, \bigcup\{W_F(x)\colon x \in G\}] \subset \mathcal{C}$ for each $G \subset F$ such that* $\operatorname{card}(G) \leq n$.

If $[F, \bigcup\{W_F(x)\colon x \in F\}] \cap [H, \bigcup\{W_H(y)\colon y \in H\}] \neq \varnothing$ for some F and H in $\mathcal{F}_{n+1}[X] - \mathcal{C}$, then each $W_F(x)$ contains exactly one element of H.

PROOF. For each $F \in \mathcal{F}_{n+1}[X] - \mathcal{C}$, let $W(F) = \bigcup\{W_F(x)\colon x \in F\}$ and suppose that $[F, W(F)] \cap [H, W(H)] \neq \varnothing$ for some F and H in $\mathcal{F}_{n+1}[X] - \mathcal{C}$. Then $F \subset W(H)$ and $H \subset W(F)$. Assume that some $W_F(x)$ contains two elements y and z of H. For each $w \in H - \{y, z\}$, let x^w be an element of F such that $w \in W_F(x^w)$. Let $G = \{x^w\colon w \in H - \{y, z\}\} \cup \{x\}$. It follows that $\operatorname{card}(G) \leq n$ and

$$G \cup H \in [G, \bigcup\{W_F(p)\colon p \in G\}] \cup [H, W(H)].$$

By (3), $[G, \bigcup\{W_F(p)\colon p \in G\}] \subset \mathcal{C}$. Hence $\mathcal{C} \cap [H, W(H)] \neq \varnothing$, which contradicts (2). Hence each $W_F(x)$ contains at most one element of H. If some $W_F(x)$ does not contain any element of H, then there exists a $W_F(p)$ which must contain at least two elements of H. From this contradiction, it follows that each $W_F(x)$ contains exactly one element of H.

THEOREM 1. *For any GO space X, the following are equivalent.*
(a) $\mathfrak{F}[X]$ *is paracompact, and*
(b) $\mathfrak{F}_2[X]$ *is paracompact.*

PROOF. We will prove the implication (b) → (a). Let X be a GO space constructed on a LOTS $(Y, <)$. For each $F \in \mathfrak{F}[X]$, we enumerate $F = \{x_1, \ldots, x_n\}$, where $x_i < x_j$ for $1 \leq i < j \leq n$. Since $\mathfrak{F}_2[X]$ is paracompact, there exists a collection $\mathfrak{U} = \{U(x)\colon x \in X\}$ of open subsets of X that weakly separates X by Lemma 2. It follows that $\mathcal{V} = \{[\{x\}, U(x)]\colon x \in X\}$ is a pairwise disjoint collection in $\mathfrak{F}[X]$ and hence, by Lemma 1, $\mathcal{V}^*$ is closed in $\mathfrak{F}[X]$ containing $\mathfrak{F}_1[X]$. Let $\mathcal{Q}$ be an open-and-closed subset of $\mathfrak{F}[X]$ such that $\mathcal{Q} \supset \mathfrak{F}_n[X]$. It suffices to prove that for each $F \in \mathfrak{F}_{n+1}[X] - \mathcal{Q}$, there exists an open set $W(F)$ containing F such that $[F, W(F)] \cap \mathcal{Q} = \varnothing$ and $\{[F, W(F)]\colon F \in \mathfrak{F}_{n+1}[X] - \mathcal{Q}\}$ is a pairwise disjoint collection.

For each $F = \{x_1, \ldots, x_{n+1}\} \in \mathfrak{F}_{n+1}[X] - \mathcal{Q}$, let $\{W_F(x_i)\colon 1 \leq i \leq n+1\}$ be a collection of order-convex sets of X such that

(1) each $W_F(x)$ is an open set containing x such that $W_F(x) \subset U(x)$,
(2) $[F, \bigcup\{W_F(x)\colon x \in F\}] \cap \mathcal{Q} = \varnothing$, and
(3) $[G, \bigcup\{W_F(x)\colon x \in G\}] \subset \mathcal{Q}$ for each $G \subset F$ such that $\operatorname{card}(G) \leq n$.

Notice that for each $x \in F$, $W_F(x) \cap F = \{x\}$. For, assume that $y \in W_F(x)$ for some $x, y \in F$ and $x \neq y$. If $G = F - \{y\}$, then $\operatorname{card}(G) = n$. Since $F \subset \bigcup\{W_F(z)\colon z \in G\}$, it follows that $F \in [G, \bigcup\{W_F(z)\colon z \in G\}]$ and hence, using (3), $F \in \mathcal{Q}$. From this contradiction, it follows that for each $x \in F$, $W_F(x) \cap F = \{x\}$. For each $F = \{x_1, \ldots, x_{n+1}\} \in \mathfrak{F}_{n+1}[X] - \mathcal{Q}$, let $W(F) = \bigcup\{W_F(x)\colon x \in F\}$. Let $\mathcal{W} = \{[F, W(F)]\colon F \in \mathfrak{F}_{n+1}[X] - \mathcal{Q}\}$. Suppose that $[F, W(F)] \cap [H, W(H)] \neq \varnothing$ for some $F = \{x_1, \ldots, x_{n+1}\}$ and $H = \{y_1, \ldots, y_{n+1}\}$ in $\mathfrak{F}_{n+1}[X] - \mathcal{Q}$. It follows that $y_i \in W_F(x_i)$ for each $i \leq n+1$. To see this, assume that $y_i \notin W_F(x_i)$ for some i. By Lemma 3, there exists exactly one element y_j of H such that $y_j \in W_F(x_i)$. Assume that $i < j$. If there exists an element y_s $(s < j)$ such that $x_i \leq y_s$, then $x_i \leq y_s < y_j$. Since $W_F(x_i)$ is order-convex, $y_s \in W_F(x_i)$. From this contradiction, it follows that $y_s < x_i$ for each $s < j$. Since $y_s < x_i < x_t$ for $s < j$ and $i < t$, it follows that

$$\{y_1, \ldots, y_j\} \subset \bigcup \{W_F(x_p)\colon p \leq i\}.$$

But, by $i < j$, there exists a $W_F(x_p)$ which contains at least two elements of H, which contradicts Lemma 3. Thus $y_i \in W_F(x_i)$, for each $i \leq n+1$. Similarly $x_i \in W_H(y_i)$ for each $i \leq n+1$. Thus, using (1), it follows that $x_i = y_i$ for each $i \leq n+1$ and $F = H$. Thus $\mathcal{W}$ is a pairwise disjoint collection. Hence $\mathfrak{F}[X]$ is paracompact.

THEOREM 2. *For any GO space X, the following are equivalent.*
(a) $\mathfrak{F}[X]$ *is metrizable, and*
(b) $\mathfrak{F}_2[X]$ *is metrizable.*

PROOF. If $\mathfrak{F}_2[X]$ is metrizable, then $\mathfrak{F}[X]$ is a Moore space [7]. By Theorem 1, $\mathfrak{F}[X]$ is a paracompact Moore space and, hence, metrizable [5]. Hence the implication (b) → (a) follows.

THEOREM 3. *Let $(Y, <)$ be a LOTS and let X be a GO space constructed on Y such that $[x, \rightarrow[$ (or $]\leftarrow, x]$) is open for each $x \in X$. Then $\mathfrak{F}[X]$ is paracompact.*

PROOF. Let $U(x) = [x, \rightarrow[$ for each $x \in X$. Then $\mathfrak{U} = \{U(x): x \in X\}$ is a collection of open subsets of X that weakly separates X. Hence $\mathfrak{F}[X]$ is paracompact by Theorem 1.

COROLLARY 1. *The Pixley-Roy hyperspace of the Sorgenfrey line is metrizable.*

A *space of ordinals* is a subspace of some ordinal.

COROLLARY 2. *Let X be a space of ordinals (resp. a first countable space of ordinals). Then $\mathfrak{F}[X]$ is paracompact (resp. metrizable). In particular, $\mathfrak{F}[[0, \omega_1)]$ is metrizable, where ω_1 is the first uncountable ordinal.*

THEOREM 4. *Let M be the Michael line. Then $\mathfrak{F}[M]$ is metrizable.*

PROOF. Since M is first countable, it suffices to prove that $\mathfrak{F}_2[M]$ is paracompact. Let $Q = \{q_n: n \in N\}$ be a counting of the rational numbers. For each $x \in M$, define an open set $U(x)$ containing x as follows.

$$U(x) = \begin{cases} M & \text{if } x = q_1, \\ M - \{q_i: i < n\} & \text{if } x = q_n \text{ and } n \geq 2, \\ \{x\} & \text{if } x \notin Q. \end{cases}$$

Then $\mathfrak{U} = \{U(x): x \in M\}$ is a collection of open subsets of M that weakly separates M. Hence $\mathfrak{F}_2[M]$ is paracompact by Lemma 2.

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