PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 1, May 1982

HOMOLOGICAL EMBEDDING PROPERTIES OF THE FIBERS OF A MAP AND THE DIMENSION OF ITS IMAGE

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ABSTRACT. A relationship is established between the homological codimension of the point inverses of a map and the dimension of its image. An infinite-dimensional version leads to the conclusion that the image of a proper map defined on Hilbert space cannot be countable dimensional. A finite-dimensional version yields: if g: $M^n \to Y$ is a proper map, M^n is a G-orientable n-manifold without boundary, and $\dim Y \leqslant k$, then there is a point $y \in Y$ and an integer $i \geqslant n-k$ such that $\check{H}^i(g^{-1}(y); G) \neq 0$.

Introduction. An argument based on a routine application of the Mayer-Vietoris sequence establishes a relationship between the homological codimension of the point inverses of a map and the dimension of its image. The motivating factor for extracting such a relationship was to establish that a proper map defined on Hilbert space cannot have an image that is countable dimensional. An analysis of this infinite-dimensional result exposed a parallel finite-dimensional version that extends a classical result for the class of dimension lowering maps defined on manifolds without boundary.

The main results are stated and proved in §2. Listed below are four corollaries; their derivations appear in §3.

COROLLARY 1. *Let g: $l_2 \to Y$ be a surjective proper map defined on a Hilbert space. Then Y is not countable dimensional.*

Properness is needed primarily to assure that each point inverse is a Z-set and, consequently, has infinite codimension (the salient feature needed in the proof). In general form, there is

COROLLARY 2. *Let g: $X \to Y$ be a surjective map from a nonempty ANR to a completely metrizable space such that each point inverse has infinite codimension in X. Then Y is not countable dimensional.*

Completeness plays an essential role even for g a proper map. Points in the subspace $l_2^f = \{(x_i) \in l_2 \colon x_i = 0$ for all but finitely many $i\}$ are Z-sets in l_2^f [4] and, therefore, the point inverses of the identity map of l_2^f have infinite codimension but l_2^f is countable dimensional.

Received by the editors July 10, 1981.

1980 *Mathematics Subject Classification*. Primary 54F45, 58B05; Secondary 54C05, 55M10, 55N05.

Key words and phrases. Z-set, infinite codimension, countable dimensional, homological embedding properties, dimension, Hilbert space, Hilbert cube.

[1] Research supported in part by NSF Grant.

0002-9939/81/0000-1120/$02.00

It is a classical result that for a proper map $g\colon X \to Y$ there is a point $y \in Y$ such that

$$\dim g^{-1}(y) \geqslant \dim X - \dim Y$$

(see [7, p. 91]). For proper maps defined on manifolds without boundary, the preceding inequality is detected by cohomology.

COROLLARY 3. *Let $g\colon M^n \to Y$ be a proper map defined on a nonempty G-orientable n-manifold without boundary and suppose that* $\dim Y \leqslant k$. *Then there is a point $y \in Y$ such that $\check{H}^i(f^{-1}(y); G) \neq 0$ for some $i \geqslant n - k$.*

By considering the restriction of f to $g^{-1}(U)$ for open subsets U of Y, it follows that the points $y \in Y$ satisfying the conclusion of the previous corollary form a dense subset of $f(Y)$.

A more discerning feature of the finite-dimensional version of the main result is illustrated by considering maps $g\colon S^k \times S^k \to I$ from the product of two k-spheres onto the interval $I = [0, 1]$. While the conclusion of Corollary 3 follows immediately since the nonendpoints separate I, the main result also yields

COROLLARY 4. *Let $g\colon S^k \times S^k \to I$ be a surjective map. Then there is a point $t \in I$ such that either $\check{H}^k(g^{-1}(t); \mathbf{Z}) \neq 0$ or $\check{H}^{k-1}(g^{-1}(t); \mathbf{Z}) \neq 0$.*

The next example illustrates that there only need be a single point t satisfying the conclusion of this last corollary for $k \geqslant 3$. The complement in $S^k \times S^k$ of the wedge $S^k \times \{\mathrm{pt}\} \cup \{\mathrm{pt}\} \times S^k$ is an open $2k$-cell and, therefore, there is an easily defined map $g\colon S^k \times S^k \to I$ such that $g^{-1}(0) = \text{point}$, $g^{-1}(1) = S^k \times \{\mathrm{pt}\} \cup \{\mathrm{pt}\} \times S^k$, and $g^{-1}(t) = S^{2k-1}$ for $0 < t < 1$.

1. Preliminaries. It is convenient to require that spaces be separable and metric. In general, the term *map* (or *mapping*) is used to specify a continuous function. A map $f\colon X \to Y$ is *proper* provided $f^{-1}(C)$ is compact for each compact subset C of Y.

The homology theory used is singular theory with coefficients in an abelian group and the cohomology theory used is Čech theory with coefficients in an abelian group. A *compact carrier* for a singular homology element $z \in H_q(U, V)$ is a pair of compacta $(P, \partial P) \subset (U, V)$ for which $z \in \operatorname{Im}\{i_*\colon H_q(P, \partial P) \to H_q(U, V)\}$. A natural source for carriers is the union of images of the maps in a singular chain representing z.

A closed subset A of an ANR (absolute neighborhood retract) X is called a *Z-set* provided the homotopy groups $\pi_*(U, U - A)$ are trivial for each open subset U of X. The closed set A is said to have *infinite codimension* provided the homology groups $H_*(U, U - A; \mathbf{Z})$ are trivial for each open subset U of X. For coefficient group G other than $\mathbf{Z}$, we shall say A has *infinite codimension with respect to the coefficient group G*. The reader is referred to [3] and [6] for further details on Z-sets and infinite codimension, respectively.

The covering dimension of a space X is denoted $\dim X$. A space is *countable dimensional* provided it is the countable union of zero-dimensional spaces. Following [7, p. 50], a space X has *small transfinite inductive dimension*, written $\operatorname{ind} X \leqslant \alpha$, provided α is an ordinal and, for each $x \in X$ and $\varepsilon > 0$, there is a neighborhood U of x having diameter $< \varepsilon$ such that $\operatorname{ind}(\operatorname{Fr} U) < \alpha$.

Amongst completely metrizable spaces, the countable-dimensional ones are precisely those that possess small transfinite inductive dimension [7, p. 51]. Consequently, neither Hilbert spaces nor the Hilbert cube possess such dimension nor does the countable-dimensional subspace $l_2^f = \{(x_i) \in l_2\colon x_i = 0$ for all but finitely many $i\}$. The reader is referred to [7] for further information on dimension theory.

2. Statements and proofs of main theorems. The main result is recorded in two forms. An essentially infinite-dimensional version that extends results in [6, §2] and a finite-dimensional version that is a "parametrized" form of a known result that is recovered in Theorem B by insisting that $K = Y$ and f be the identity (see [5, Appendix]). In all cases, the critical part of the proof is a routine Mayer-Vietoris argument.

THEOREM A. *Let Y be a space having small transfinite inductive dimension, let K be a closed subset of an ANR X, and let $f\colon K \to Y$ be a map with each point-inverse $f^{-1}(y)$ having infinite codimension in X with respect to a coefficient group G. Then K has infinite codimension in X with respect to the coefficient group G.*

THEOREM B. *Let Y be a finite-dimensional space, let K be a closed subset of an ANR X, let $f\colon K \to Y$ be a map, and let q be an integer such that $H_i(X, X - f^{-1}(y); G) = 0$ for each $y \in Y$ and $q \leq i \leq q + \dim Y$. Then $H_q(X, X - K; G) = 0$.*

Both proofs proceed by induction on the dimension of Y, the induction being transfinite in one case. The slightly more difficult proof of Theorem B is presented next while the easy task of extracting a proof of Theorem A is left to the reader. The coefficient group G is suppressed.

The conclusion is trivially satisfied for $\dim Y = -1$ as K must be empty. Inductively, we assume not only that $H_q(X, X - f^{-1}(L)) = 0$ for each closed subset $L \subset Y$ with $\dim L < \dim Y$, but also that $H_{q+1}(X, X - f^{-1}(L)) = 0$, since $H_i(X, X - f^{-1}(y)) = 0$ for each $y \in L$ and $q \leq i \leq q + 1 + \dim L$.

The remainder of the argument uses the observation that, for a pair of closed subsets A and B of X, the Mayer-Vietoris sequence for "the excisive couple of pairs" $\{(X, X - A), (X, X - B)\}$ yields an inclusion induced isomorphism

$$H_q(X, X - (A \cup B)) \to H_q(X, X - A) \oplus H_q(X, X - B)$$

whenever the adjacent terms $H_q(X, X - (A \cap B))$ and $H_{q+1}(X, X - (A \cap B))$ are trivial.

Fix $z \in H_q(X, X - K)$ and choose a compact carrier $(P, \partial P)$ for z. Specify a neighborhood V_y of each $y \in f(P \cap K)$ such that the image of z in $H_q(X, X - f^{-1}(V_y))$ is trivial; this is possible since by hypothesis $H_q(X, X - f^{-1}(y)) = 0$ and, therefore, z has a *compact* carrier missing $f^{-1}(y)$. Refine the cover $\{V_y\}$ of $f(P \cap K)$ by a cover $\{R_i\colon i = 1, 2, \ldots, k\}$ of $f(P)$ by closed sets such that the interior of $\bigcup_{i=1}^{k} R_i$ contains $f(P \cap K)$ and the frontier of each R_i has dimension strictly less than the dimension of Y.

Define sets $A_j = Cl(Y - \cup_{i=j+1}^{k} R_i)$ where $j = 0, 1, \ldots, k$. Since A_0 does not intersect $f(P \cap K)$, the image of z in $H_q(X, X - f^{-1}(A_0))$ is trivial. Inductively, for $j = 1, 2, \ldots, k$, we assume that the image of z in $H_q(X, X - f^{-1}(A_{j-1}))$ is trivial and, by choice, its image is trivial in $H_q(X, X - f^{-1}(R_j))$. Since $f^{-1}(A_{j-1}) \cap f^{-1}(R_j) = f^{-1}(A_{j-1} \cap R_j)$ and $\dim A_{j-1} \cap R_j < \dim Y$ as $A_{j-1} \cap R_j \subset \operatorname{Fr} R_j$, we have inductively that $H_e(X, X - (f^{-1}(A_{j-1}) \cap f^{-1}(R_j)))$ is trivial for $e = q$, $q + 1$ and the Mayer-Vietoris argument given earlier yields an isomorphism

$$H_q\big(X, X - \big(f^{-1}(A_{j-1}) \cup f^{-1}(R_j)\big)\big) \to H_q\big(X, X - f^{-1}(A_{j-1})\big) \oplus H_q\big(X, X - f^{-1}(R_j)\big).$$

We conclude that the image of z in $H_q(X, X - f^{-1}(A_j))$ is trivial. Finally, when $j = k$ we see that z itself is trivial.

3. Proofs of corollaries.

PROOFS OF COROLLARIES 1 AND 2. Since the image of a proper map defined on a complete metric space is completely metrizable [9], the first corollary follows from the second as compact subsets of l_2 are Z-sets [2]. Corollary 2 is proved by way of contradiction. If Y were countable dimensional, then Y would have small transfinite inductive dimension [7, p. 51] and Theorem A would apply with $G = \mathbf{Z}$ and $K = X$ and would yield that X has infinite codimension in X. This is not possible since $H_0(X, X - X; \mathbf{Z}) = H_0(X; \mathbf{Z}) \neq 0$.

PROOF OF COROLLARY 3. There is a point $y \in Y$ and an integer $0 < i < \dim Y$ such that $H_i(M, M - f^{-1}(y); G) \neq 0$; otherwise, Theorem B would apply with $K = M$ and $q = 0$ and would contradict that $H_0(M, M - M; G) = H_0(M; G) \neq 0$. Alexander duality [8, p. 296] yields that $\check{H}^{n-i}(f^{-1}(y); G) \neq 0$ and it is transparent that $n - i \geqslant n - \dim Y \geqslant n - k$.

PROOF OF COROLLARY 4. In order to avoid being able to use Theorem B with $K = X$, $G = \mathbf{Z}$, and $q = k$ and contradicting that $H_k(S^k \times S^k; \mathbf{Z}) \neq 0$, there must be a point $t \in I$ such that $H_e(S^k \times S^k, S^k \times S^k - f^{-1}(y); \mathbf{Z}) \neq 0$ for one of $e = k$, $k + 1$. Alexander duality yields that either $\check{H}^k(f^{-1}(y); \mathbf{Z}) \neq 0$ or $\check{H}^{k-1}(f^{-1}(y); \mathbf{Z}) \neq 0$.

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